

Twirling of hula-hoop: new results

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Summary. We consider twirling of a hula-hoop when the waist of a gymnast moves along an elliptic trajectory close to a circle. For the case of the circular trajectory, two families of exact solutions are obtained. Both of them correspond to twirling of the hula-hoop with a constant angular speed equal to the speed of the excitation. We show that one family of solutions is stable, while the other one is unstable. These exact solutions allow us to obtain approximate solutions for the case of an elliptic trajectory of the waist. We demonstrate that to twirl a hula-hoop one needs to rotate the waist with a phase difference lying between $\pi/2$ and π . An interesting effect of inverse twirling is described when the waist moves in a direction opposite to the hula-hoop rotation. The approximate analytical solutions are compared with the results of numerical simulation.

Introduction

A hula-hoop is a popular toy – a thin hoop that is twirled around the waist, limbs or neck. In recent decades it is widely used as an implement for fitness and gymnastic performances, Fig. 1.¹ To twirl a hula-hoop the waist of a gymnast carries out a periodic motion in the horizontal plane. For the sake of simplicity we consider the two-dimensional problem disregarding the vertical motion of the hula-hoop. We assume that the waist is a circle and its center moves along an elliptic trajectory close to a circle.

Previously considered was the simple case in which a hula-hoop is treated as a pendulum with the pivot oscillating along a line, see [1, 2]. The stationary rotations of a hula-hoop excited in two directions have been studied by an approximate method of separate motions in [3]. The similar problem of the spinner mounted loosely on a pivot with a prescribed bi-directional motion has been treated numerically and experimentally in [4].

Here we derive the exact solutions in the case of a circular trajectory of the waist center and approximate solutions in the case of an elliptic trajectory. We also check the condition of keeping contact with the waist during twirling. The paper differs from our previous one [5] by new approximate solutions found analytically and compared with the numerical simulation results.



Figure 1: Viktoria Bagrova in the Nikulin Circus performance, Moscow 2008 (photo by K. Stukalov).

Main relations

We assume that the center O' of a gymnast's waist moves in time according to the elliptic law $x = a \sin \omega t$, $y = b \cos \omega t$ with the amplitudes a , b and the excitation frequency $\omega > 0$, Fig. 2. The equations of motion in the waist-fixed coordinate

¹The same model lies in the basis of some industrial machinery such as vibrating cone crushers designed for crushing hard brittle materials, see [3].

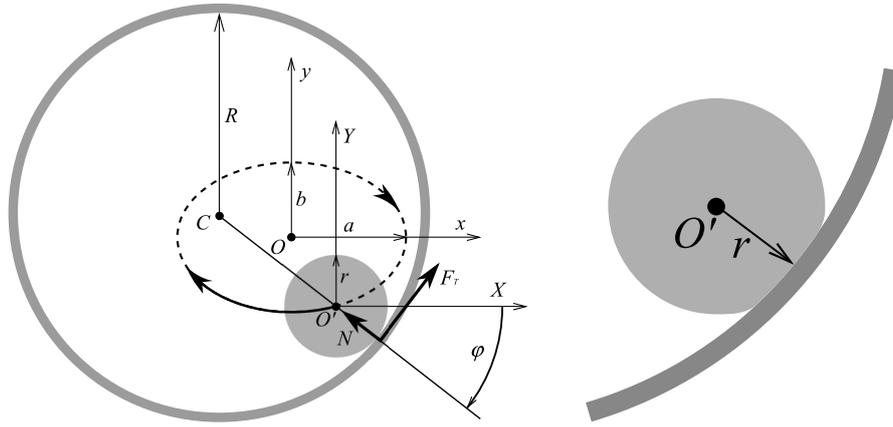


Figure 2: A hula-hoop with the radius R twirling with the angle φ around a circular waist (shaded) with the radius r . The center O' of the waist moves along the elliptic curve $x = a \sin \omega t$, $y = b \cos \omega t$ with the fixed center O . The hula-hoop acts on the waist with normal force N and tangential friction force F_T . There is also a rolling resistance due to the waist deformation (right).

system take the following form

$$I_C \ddot{\theta} + k \dot{\theta} = -F_T R - d N \text{sign}(\dot{\theta}) \quad (1)$$

$$m(R-r)\ddot{\varphi} = m(\ddot{x} \sin \varphi + \ddot{y} \cos \varphi) + F_T \quad (2)$$

$$m(R-r)\dot{\varphi}^2 = N + m(\ddot{x} \cos \varphi - \ddot{y} \sin \varphi) \quad (3)$$

where θ is the rotation angle around center of mass C , $I_C = mR^2$ is the central moment of inertia of the hula-hoop, φ is the angle between axis x and radius CO' , r is the radius of the waist, m and R are the mass and radius of the hula-hoop. Equation (1) describes change of angular momentum due to linear viscous damping with coefficient k , rolling drag (rolling resistance) with coefficient d , and the tangential friction force F_T between the waist and the hoop. Equations (2) and (3) describe the motion of the hula-hoop in the longitudinal and transverse directions to the radius CO' , where N is the normal reaction force of the hula-hoop to the waist. Equations (2) and (3) contain additional inertial forces since the waist-fixed reference system is noninertial.

Assuming that slipping at the point of contact is absent we obtain the kinematic relation

$$(R-r)\dot{\varphi} = R\dot{\theta} \quad (4)$$

We exclude from equations (1) and (2) the force F_T and with relation (4) obtain the equation of motion

$$\ddot{\varphi} + \frac{k}{2mR^2}\dot{\varphi} + \frac{d}{2R} \text{sign}\dot{\varphi} \left(\dot{\varphi}^2 + \frac{\omega^2 (a \sin \omega t \cos \varphi - b \cos \omega t \sin \varphi)}{R-r} \right) + \frac{\omega^2 (a \sin \omega t \sin \varphi + b \cos \omega t \cos \varphi)}{2(R-r)} = 0. \quad (5)$$

From equation (3) we find the normal force and imply the condition $N > 0$ as

$$(R-r)\dot{\varphi}^2 + \omega^2 (a \sin \omega t \cos \varphi - b \cos \omega t \sin \varphi) > 0 \quad (6)$$

which means that the hula-hoop during its motion keeps contact with the waist of the gymnast.

We introduce new time $\tau = \omega t$ and non-dimensional parameters

$$\gamma = \frac{k}{2mR^2\omega}, \quad \delta = \frac{d}{2R}, \quad \varepsilon = \frac{a-b}{4(R-r)}, \quad \mu = \frac{a+b}{4(R-r)}, \quad (7)$$

the damping coefficient γ , the rolling resistance coefficient δ , and excitation parameters μ and ε . Relation between μ and ε determines the form of ellipse – the trajectory of the waist center. For $\varepsilon = \mu$ the trajectory is a line, and for $\varepsilon = 0$ it is a circle. Then equation (5) and inequality (6) take the form

$$\ddot{\varphi} + \gamma \dot{\varphi} + \delta \dot{\varphi} |\dot{\varphi}| + \mu \cos(\varphi - \tau) - 2\mu \delta \text{sign}(\dot{\varphi}) \sin(\varphi - \tau) = \varepsilon \cos(\varphi + \tau) - 2\varepsilon \delta \text{sign}(\dot{\varphi}) \sin(\varphi + \tau) \quad (8)$$

$$\dot{\varphi}^2 - 2\mu \sin(\varphi - \tau) + 2\varepsilon \sin(\varphi + \tau) > 0 \quad (9)$$

where the dot means differentiation with respect to the time τ .

Exact solutions

When the waist center moves along a circle ($a = b$, i.e. $\varepsilon = 0$) equation (8) takes the following form

$$\ddot{\varphi} + \gamma \dot{\varphi} + \delta \dot{\varphi} |\dot{\varphi}| + \mu \cos(\varphi - \tau) - 2\mu \delta \text{sign}(\dot{\varphi}) \sin(\varphi - \tau) = 0 \quad (10)$$

and has the exact solution [5]

$$\varphi = \tau + \varphi_0 \quad (11)$$

with the constant initial phase φ_0 given by the equation

$$\gamma + \delta + \mu \cos \varphi_0 - 2\mu\delta \sin \varphi_0 = 0. \quad (12)$$

Therefore, solution (11) exists only under the condition $|\gamma + \delta| \leq |\mu|\sqrt{1 + 4\delta^2}$, so we find from equation (12)

$$\varphi_0 + \arccos\left(\frac{1}{\sqrt{1 + 4\delta^2}}\right) = \pm \arccos\left(-\frac{\gamma + \delta}{\mu\sqrt{1 + 4\delta^2}}\right) + 2\pi n, \quad n = 1, 2, \dots \quad (13)$$

provided that $\mu \neq 0$. Solutions (11), (13) correspond to the rotation of the hula-hoop with the constant angular velocity equal to the excitation frequency ω .

Stability of the exact solutions

Let us investigate the stability of the obtained solutions. For this purpose we take the angle φ in the form $\varphi = \tau + \varphi_0 + \eta(\tau)$ where $\eta(\tau)$ is a small quantity, and substitute it into equation (10). Taking linearization with respect to η and with the use of (12) we obtain a linear equation

$$\ddot{\eta} + (\gamma + 2\delta)\dot{\eta} - \mu(\sin \varphi_0 + 2\delta \cos \varphi_0)\eta = 0. \quad (14)$$

According to Lyapunov's theorem on the stability based on a linear approximation [6] solution (11), (13) is asymptotically stable if all the eigenvalues of linearized equation (14) have negative real parts. From Routh-Hurwitz criterion [6] we obtain the stability conditions as

$$\gamma + 2\delta > 0, \quad \mu(\sin \varphi_0 + 2\delta \cos \varphi_0) < 0. \quad (15)$$

Without loss of generality we assume $\mu > 0$ since the case $\mu < 0$ can be reduced to the previous one by the time transformation $\tau' = \tau + \pi$ in equation (8). The second condition in (15) can be written as $\sin(\varphi_0 + \arccos(1/\sqrt{1 + 4\delta^2})) < 0$. Thus, from conditions (15), relation (13) and due to the assumption $\mu > 0$ we find that for $0 < \gamma + 2\delta < \delta + \mu\sqrt{1 + 4\delta^2}$ solution (11) with

$$\varphi_0 = -\arccos\left(-\frac{\gamma + \delta}{\mu\sqrt{1 + 4\delta^2}}\right) - \arccos\left(\frac{1}{\sqrt{1 + 4\delta^2}}\right) + 2\pi n, \quad n = 1, 2, \dots \quad (16)$$

is asymptotically stable, and solution (11) with

$$\varphi_0 = \arccos\left(-\frac{\gamma + \delta}{\mu\sqrt{1 + 4\delta^2}}\right) - \arccos\left(\frac{1}{\sqrt{1 + 4\delta^2}}\right) + 2\pi n, \quad n = 1, 2, \dots \quad (17)$$

is unstable.

Condition for hula-hoop's contact with the waist

Let us verify for the exact solutions (11), (13) the condition of twirling without losing contact (9) which takes the form

$$\mu \sin \varphi_0 < \frac{1}{2}. \quad (18)$$

The second stability condition in (15) can be rewritten with the use of (12) as follows

$$\mu \sin \varphi_0 < \frac{2\delta(\gamma + \delta)}{1 + 4\delta^2}.$$

Thus, stable solution (11), (16) provides asymptotically stable twirling of the hula-hoop with the constant angular velocity ω without losing contact with the waist of the gymnast under the condition $\gamma\delta < 1/4$.

In the case of without rolling resistance $\delta = 0$ condition (18) is always satisfied for the stable solution (11), (16). While for unstable solution (11), (17) we have $\mu \sin \varphi_0 = \sqrt{\mu^2 - \gamma^2}$ so condition (18) holds only if $\mu < \sqrt{1/4 + \gamma^2}$. The phase φ_0 of the stable solution belongs to the interval $[-\pi, -\pi/2] \bmod 2\pi$, and for vanishing damping $\gamma \rightarrow +0$ the phase tends to $-\pi/2$. Below we will show that this phase inequality also holds for the approximate solutions. This is how to twirl a hula-hoop!

Approximate solutions

Let us find approximate solutions for the case of close but not equal amplitudes $a \approx b$. For the sake of simplicity from now on we will keep $\delta = 0$ and assume that $a \geq |b|$ which means $\varepsilon \geq 0$, $\mu \geq 0$. Taking ε as a small parameter we apply perturbation method assuming that the exact solution $\varphi_s(\tau)$ of (8) can be expressed in a series

$$\varphi_s(\tau) = \tau + \varphi_0 + \varepsilon\varphi_1(\tau) + o(\varepsilon). \quad (19)$$

After substitution of series (19) in (8) and grouping the terms by equal powers of ε we derive the following chain of equations

$$\varepsilon^0: \quad \gamma + \mu \cos(\varphi_0) = 0 \quad (20)$$

$$\varepsilon^1: \quad \ddot{\varphi}_1 + \gamma \dot{\varphi}_1 - \mu \sin(\varphi_0) \varphi_1 = \cos(\varphi_0 + 2\tau) \quad (21)$$

Taking solution of equation (20) for $\mu > 0$

$$\varphi_0 = -\arccos(-\gamma/\mu) + 2\pi n, \quad n = 1, 2, \dots \quad (22)$$

corresponding to the stable unperturbed solution (11), (16) we write equation (21) as

$$\ddot{\varphi}_1 + \gamma \dot{\varphi}_1 + \sqrt{\mu^2 - \gamma^2} \varphi_1 = \cos(\varphi_0 + 2\tau) \quad (23)$$

It has a unique periodic solution

$$\varphi_1(\tau) = C \sin(2\tau + \varphi_0) + D \cos(2\tau + \varphi_0) \quad (24)$$

where constants C and D defined as follows

$$C = \frac{2\gamma}{\mu^2 + 3\gamma^2 - 8\sqrt{\mu^2 - \gamma^2} + 16}, \quad D = \frac{-4 + \sqrt{\mu^2 - \gamma^2}}{\mu^2 + 3\gamma^2 - 8\sqrt{\mu^2 - \gamma^2} + 16}. \quad (25)$$

We see that the approximate solution $\varphi(\tau) = \tau + \varphi_0 + \varepsilon \varphi_1(\tau)$ with (22), (24), (25) differs from the exact solution (11), (16) of the unperturbed system by small vibrating terms of frequency 2, see (24). Note that the approximate solutions were obtained with the assumption that the excitation amplitudes and damping are not small.

Stability of the approximate solutions

To find the stability conditions for solution (19), (22), (24) we take a small variation to the solution of (19) $\varphi = \varphi_s + u$ and substitute this expression into (8). After linearization with respect to u and keeping only terms of first order we obtain a linear equation

$$\ddot{u} + \gamma \dot{u} + (-\mu(\sin \varphi_0 + \varepsilon \varphi_1 \cos \varphi_0) + \varepsilon \sin(2\tau + \varphi_0)) u = 0 \quad (26)$$

where φ_0 is given by expression (22). Equation (26) can be written in the form of damped Mathieu-Hill equation as

$$\ddot{u} + \gamma \dot{u} + (p + \varepsilon \Phi(2\tau)) u = 0 \quad (27)$$

where $p = -\mu \sin \varphi_0 = \sqrt{\mu^2 - \gamma^2}$, $\Phi(2\tau) = (\gamma C + 1) \sin(2\tau + \varphi_0) + \gamma D \cos(2\tau + \varphi_0)$. Then the stability condition (absence of parametric resonance at all frequencies \sqrt{p}) is given by the inequalities [6]

$$\varepsilon < \frac{2\gamma}{\sqrt{(\gamma C + 1)^2 + \gamma^2 D^2}} \quad (28)$$

with C and D defined in (25). This is the inequality to the problem parameters γ , ε and μ .

Condition for hula-hoop's contact with the waist

The condition of twirling without losing contact (9) takes the following form

$$\varepsilon < \frac{1 + 2\sqrt{\mu^2 - \gamma^2}}{2} \sqrt{\frac{\mu^2 + 3\gamma^2 - 8\sqrt{\mu^2 - \gamma^2} + 16}{\mu^2 + 8\gamma^2 - 12\sqrt{\mu^2 - \gamma^2} + 36}}. \quad (29)$$

Conditions (28), (29) imply restrictions to ε , i.e. how much the elliptic trajectory of the waist center differs from the circle.

Comparison with numerical simulations

In Fig. 3 the approximate analytical solution is presented and compared with the results of numerical simulation for the case when the excitation parameter μ and damping coefficient γ are not small.

Small excitation amplitudes and damping

It is interesting to consider the case when the excitation amplitudes and damping coefficient are small having the same order as ε . Then we introduce new parameters $\tilde{\mu} = \mu/\varepsilon$ and $\tilde{\gamma} = \gamma/\varepsilon$ and assume that the solution has the form

$$\varphi(\tau) = \rho\tau + \varphi_0(\tau) + \varepsilon \varphi_1(\tau) + o(\varepsilon), \quad (30)$$

where ρ is the angular velocity of rotation, and the functions $\varphi_0(\tau)$, $\varphi_1(\tau)$ are supposed to be bounded.

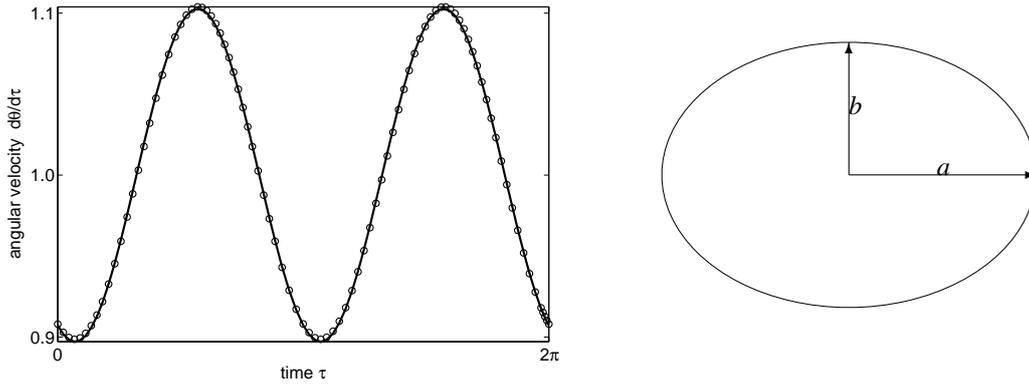


Figure 3: Comparison between the approximate analytical (solid line) and numerical (circles) results, for parameters: $\mu = 1.2$, $\varepsilon = 0.2$, and $\gamma = 1$. Initial conditions for numerical calculations are $\theta(0) = -2.5389$, $\dot{\theta}(0) = 0.90802$. Proportions of the waist center trajectory are presented by the ellipse on the right.

We substitute expression (30) into (8) and equating terms of the same powers of ε obtain the following equations

$$\varepsilon^0: \quad \ddot{\varphi}_0 = 0 \quad (31)$$

$$\varepsilon^1: \quad \ddot{\varphi}_1 = \cos(\varphi_0 + \tau + \rho\tau) - \tilde{\mu} \cos(\varphi_0 - \tau + \rho\tau) - \tilde{\gamma} \dot{\varphi}_0 - \tilde{\gamma} \rho. \quad (32)$$

From equation (31) we get that function $\varphi_0(\tau)$ can remain bounded only if it is constant ($\varphi_0(\tau) \equiv \varphi_0 = \text{const}$). Then equation (32) can have bounded solutions $\varphi_1(\tau)$ only when ρ takes the values -1, 0, 1. Thus, besides clockwise rotation we have also counterclockwise rotation $\rho = -1$, and no rotational solution $\rho = 0$. The latter is not interesting, so we omit it.

Clockwise rotation

For clockwise rotation $\rho = 1$ in Fig. 4 a) from equations (30), (31), and (32) we obtain in the first approximation the solution

$$\varphi_*(\tau) = \tau + \varphi_0 + \varepsilon \varphi_1(\tau), \quad (33)$$

$$\varphi_1(\tau) = -\frac{1}{4} \cos(\varphi_0 + 2\tau), \quad \cos \varphi_0 = -\frac{\gamma}{\mu}, \quad (34)$$

where function $\varphi_1(\tau)$ is found up to the addition of a constant, which we have set to zero for determinacy. Thus, we let only φ_0 contain a constant term of the solution. The first expression in (34) is the special case of (24), (25) when $\mu = \gamma = 0$ in (25).

To verify the stability conditions for solution (33) we use damped Mathieu-Hill equation (27) and for the case of small damping and excitation amplitudes get $\gamma > 0$, $\sin \varphi_0 < 0$, see [6]. These conditions are similar with inequalities (15) derived for the undisturbed exact solution. Thus, the stable solution (33) with $\varphi_0 = -\arccos(-\gamma/\mu) + 2\pi n$ exists for

$$0 < \gamma < \mu. \quad (35)$$

For solution (33) condition (9) of keeping contact in the first approximation reads

$$\varepsilon < \frac{1 + 2\sqrt{\mu^2 - \gamma^2}}{3}, \quad (36)$$

and holds true for sufficiently small ε .

Counterclockwise rotation

For counterclockwise rotation $\rho = -1$ in Fig. 4 b) we obtain in the first approximation the solution

$$\varphi_*(\tau) = -\tau + \varphi_0 + \frac{\mu}{4} \cos(\varphi_0 - 2\tau), \quad \cos \varphi_0 = -\frac{\gamma}{\varepsilon}, \quad (37)$$

with the stability conditions $\gamma > 0$, $\sin \varphi_0 > 0$. Thus, the stable counterclockwise rotation (37) with $\varphi_0 = \arccos(-\gamma/\varepsilon) + 2\pi n$ exists for

$$0 < \gamma < \varepsilon. \quad (38)$$

For this case condition (9) takes the form similar to (36) and holds true for sufficiently small μ

$$\mu < \frac{1 + 2\sqrt{\varepsilon^2 - \gamma^2}}{3}. \quad (39)$$

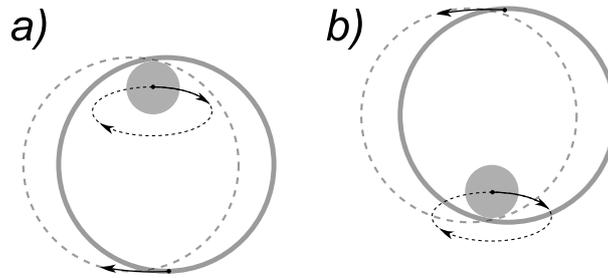


Figure 4: Stable twirling of the hula-hoop for the cases: a) direct twirling b) inverse twirling.

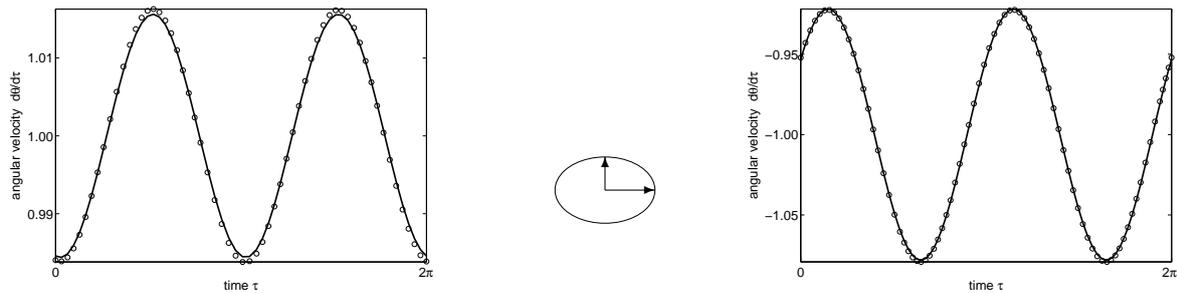


Figure 5: Comparison between the approximate analytical (solid line) and numerical (circles) results for small excitation amplitudes and damping coefficient for clockwise (left) and counterclockwise (right) rotations, for parameters: $\mu = 5/32$, $\varepsilon = 1/32$, and $\gamma = 1/40$. Initial conditions for numerical calculations are $\theta(0) = -1.7301$, $\dot{\theta}(0) = 0.98403$ (left) and $\theta(0) = 2.4696$, $\dot{\theta}(0) = -0.95212$ (right). The exact proportions of the waist center trajectory are depicted in the middle.

Coexistence of clockwise and counterclockwise rotations

It follows from conditions (35), (38) that stable clockwise and counterclockwise rotations (33), (34) and (37) coexist if the following conditions are satisfied

$$0 < \gamma < \min\{\varepsilon, \mu\}. \quad (40)$$

Conditions (40) in physical variables take the form

$$0 < 2k \frac{R-r}{R^2 \omega m} < a - |b|, \quad (41)$$

meaning that the trajectory of the waist should be sufficiently prolate. Coexisting clockwise and counterclockwise rotations are illustrated in Fig. 4.

Comparison with numerical simulations

In Fig. 5 the approximate analytical solutions for rotations in both directions are presented and compared with the results of numerical simulation for the case of small excitation parameters μ , ε and the damping coefficient γ . The values of μ , ε correspond to the dimensional parameters $a = 15\text{cm}$, $b = 10\text{cm}$, $r = 10\text{cm}$, $R = 50\text{cm}$.

Conclusion

We have derived simple explicit relations showing how to twirl a hula-hoop stably in both directions without losing contact with the waist. The approximate analytical solutions are in a good agreement with the results of numerical simulation for plausible parameters of the problem. The model considered can have not only academic but also industrial applications.

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