Another mechanical model of parametrically excited pendulum and stabilization of its inverted equilibrium position

Anton O. Belyakov

ORCOS, Institute of Mathematical Methods in Economics, Vienna University of Technology, Austria
Institute of Mechanics, Lomonosov Moscow State University, Russia

Summary. A new mechanical model of parametrically excited pendulum is proposed. The pendulum consists of two identical masses vibrating symmetrically along a circle. The model is mathematically equivalent to the pendulum with vertically vibrating pivot. Hence, the same parametric resonances and dynamic regimes can be observed. But the new mechanical realization provides different explanations of e.g. such phenomenon as high frequency stabilization of the inverse vertical position. Effect of damping on stabilization of the inverse vertical position is studied. It is observed that increase of damping can shift the stability domain of the inverse vertical position, i.e. damping can both stabilize and destabilize this equilibrium position. Similarity between the stability domains for harmonic and piecewise-constant periodic excitation functions for small parameters is demonstrated.

Schemes of equivalent parametric pendula

We consider the pendulum presented in Fig. 1 which is governed by the same dimensionless equation as the pendulum with vertically vibrating pivot in Fig. 2, see [1, 2] and references therein. Thus, we have the same plethora of dynamic regimes such as oscillations, rotations, oscillation-rotations, and tumbling chaos. Stability properties of two vertical positions ($\theta = 0$ and $\theta = \pi$) of the pendulum are described by the same *Ince-Strutt diagram*, see Fig. 3. Stability regions of the pendulum in Fig. 2 with sinusoidal excitation function were studied both with and without damping, see [2] and [3] correspondingly, and the references therein. The effect of stabilization of the inverted vertical position was intensely studied, e.g. [1, 2]. The stability analysis can be simplified when the periodic excitation function is piecewise-constant (solid line in the top of Fig 8), as described in §28, Chapter 3 in [4]. For the pendulum in Fig. 1 such excitation would involve sharp changes of the angel 2ψ between the the rods, while for the pendulum in Fig. 2 the excitation force, applied to its pivot, changes sharply so that the pivot coordinate y moves piecewise-parabolically in time (solid line in the bottom of Fig 8).

Pendulum scheme in Fig. 1 has the center of mass close to the pivot, see e.g. [5]. We shift the center of mass periodically keeping the same moment of inertia with respect to the pivot of the pendulum. Thus, we change the eigenfrequency of small undamped oscillations of the pendulum, reproducing the same effect on the eigenfrequency as can be done by the vertical oscillations of the pivot in Fig. 2. Therefore, we obtain formally the same dimensionless equation with parametric excitation for both pendular. Nevertheless, from the physical point of view, the dimensionless parameter for excitation amplitude should be chosen differently. The high-frequency stabilization of the inverse vertical position is impossible for the pendulum in Fig. 1 when $l_a < l_0$.

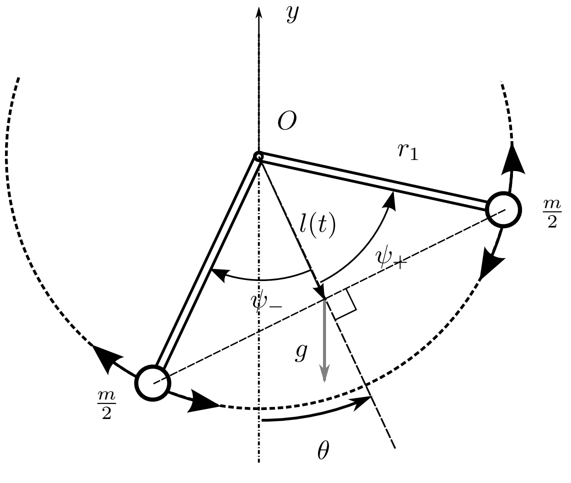

Figure 1: Parametric pendulum with shifting center of mass

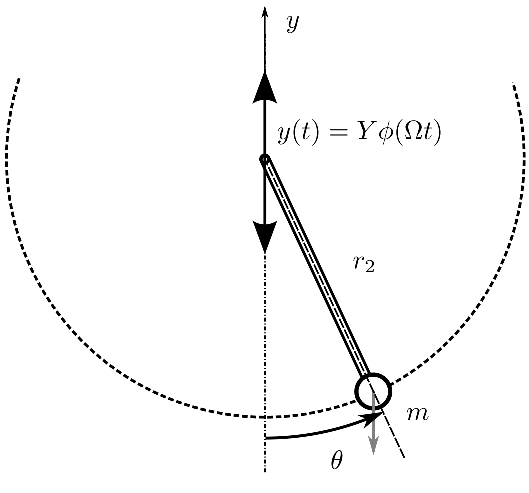

Figure 2: Parametric pendulum with vibrating support

Notice that not all parametrically excited pendula can be described by the same equation, as e.g. pendulum with periodically varying (strictly positive) length has no stabilization of the inverse vertical position, see Section 9 in [6].

Equations of motion

Polar coordinates of material points with respect to the pendulum symmetry line are $(\psi_\pm(t), r_1)$, where

$$\psi_\pm(t) = \pm \arccos\left(\frac{l_0 + l_a\varphi(\Omega t)}{r_1}\right) \tag{1}$$

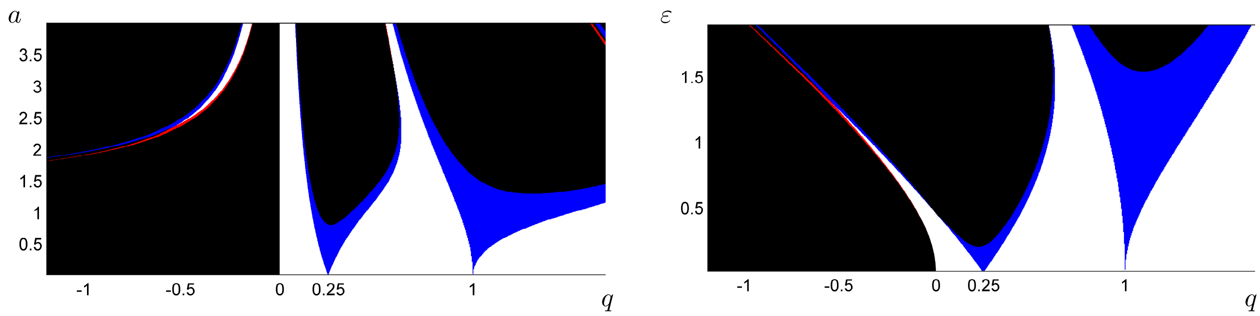

Figure 3: Ince-Strutt diagram for $\varphi(\tau) = \cos(\tau)$ depicts ($q > 0$) the stability of the vertical pendulum position, $\theta = 0$, and ($q < 0$) inverted vertical position, $\theta = \pi$, depending on relative eigenfrequency $\omega = \sqrt{|q|}$ and relative excitation amplitude a (left) or ε (right), defined in (3) and (6). Left and right diagrams are the same up to the transformation $\varepsilon = a\omega^2$. Black color depicts the resonance domains for both $\beta = 0.0001$ and $\beta = 0.4$. Blue color marks the domains of damping stabilization with $\beta = 0.4$, while red denotes the domain where damping $\beta = 0.4$ destabilizes the vertical position ($\theta = 0$ or $\theta = \pi$).

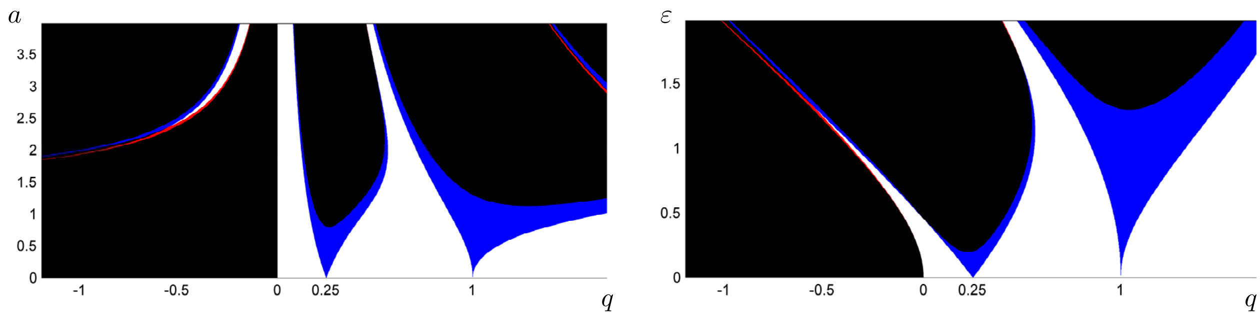

Figure 4: Ince-Strutt diagram for stepwise excitation function $\varphi(\tau) = \frac{\pi}{4}\,\mathrm{sign}(\cos(\tau))$ depicts ($q > 0$) the stability of the vertical pendulum position, $\theta = 0$, and ($q < 0$) inverted vertical position, $\theta = \pi$, depending on relative eigenfrequency $\omega = \sqrt{|q|}$ and relative excitation amplitude a (left) or ε (right), defined in (3) and (6). Left and right diagrams are the same up to the transformation $\varepsilon = a\omega^2$. Black color depicts the resonance domains for both $\beta = 0.0001$ and $\beta = 0.4$. Blue color marks the domains of damping stabilization with $\beta = 0.4$, while red denotes the domain where damping $\beta = 0.4$ destabilizes the vertical position ($\theta = 0$ or $\theta = \pi$).

with zero mean 2π-periodic *excitation function* $\varphi(\tau) = \varphi(\tau + 2\pi)$ and $|l_0 + l_a\varphi(\Omega t)| \le r_1$ for all τ; r_1 is the radius of the pendulum, l_0 is the average position of the mass center. Moment of inertia remains constant $I = mr_1^2$, while the center of mass $l(t) = r_1 \cos(\psi_\pm(t)) = l_0 + l_a\varphi(\Omega t)$ moves along the symmetry line of the pendulum. The equation of motion is derived with the use of angular momentum alteration theorem and taking into account linear damping forces

$$mr_1^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c_1 \frac{\mathrm{d}\theta}{\mathrm{d}t} + m\,g\,l(t)\sin(\theta) = 0 \tag{2}$$

where m is the mass, c is the viscous damping coefficient, l is the distance between the pivot and the center of gravity, θ is the angle of the pendulum symmetry line deviation from the vertical position, t is time, g is gravitational acceleration.

Dimensionless equation

We introduce new time $\tau = \Omega t$ and three dimensionless parameters a, ω, β, along with the eigenfrequency Ω_0 as follows

$$a = \frac{l_a}{l_0}, \quad \omega = \frac{\Omega_0}{\Omega}, \quad \beta = \frac{c_1}{\Omega_0 m r_1^2}, \quad \Omega_0 = \frac{\sqrt{l_0\, g}}{r_1}, \tag{3}$$

where Ω_0 is the frequency of small oscillations when $l_a = 0$ and $c = \beta = 0$. Dimensionless equation (2) takes the form

$$\ddot{\theta} + \beta\omega\dot{\theta} + \omega^2\left(1 + a\varphi(\tau)\right)\sin(\theta) = 0, \tag{4}$$

where the upper dot denotes differentiation with respect to new time τ. Next, we show that this equation is equivalent to the equation of the pendulum with vertically vibrating support.

Conditions for the same equation of motion

$$\varphi(\tau) = -\ddot{\phi}(\tau), \quad \frac{l_a}{l_0} = \frac{\Omega^2 Y}{g}, \quad \frac{c_1}{r_1^2} = \frac{c_2}{r_2^2}, \tag{5}$$

where $l_0 + l_a\varphi(\Omega t) \le r_1$ for all τ. In order to have the same eigenfrequency Ω_0 with the same total mass of the pendulum m we need to require that $r_1^2 = r_2\, l_0$. Three dimensionless parameters ε, ω, β, with the same eigenfrequency Ω_0

$$\varepsilon = \frac{Y}{r_2}, \quad \omega = \frac{\Omega_0}{\Omega}, \quad \beta = \frac{c_2}{\Omega_0 m r_2^2}, \quad \Omega_0 = \sqrt{\frac{g}{r_2}}, \tag{6}$$

yield in the time $\tau = \Omega t$ the equation

$$\ddot{\theta} + \beta\omega\dot{\theta} + \big(\omega^2 + \varepsilon\varphi(\tau)\big)\sin(\theta) = 0, \tag{7}$$

which coincides with that of a pendulum with vibrating support (4), provided that $\varepsilon = a\,\omega^2$. When the excitation function $\varphi(\tau)$ is sinusoidal, $\varphi(\tau) = \sin(\tau)$, we have the well known Ince-Strutt diagram depicting instability (parametric resonance) domains, see Fig. 3. We see, that there is no high frequency stabilization of the inverted position, $\theta = \pi$, in parameter space (q, a) in contrast to (q, ε). But there is still possibility of "low frequency stabilization" for sufficiently large $a > 1$, such that $l_a > l_0$.

Stability analysis

Let us perturbed solutions of equations (4) and (7) by small η, then we have the corresponding linearized equations:

$$\ddot{\eta} + \beta\omega\dot{\eta} \pm \omega^2\,(1 + a\varphi(\tau))\,\eta = 0, \qquad \ddot{\eta} + \beta\omega\dot{\eta} \pm \big(\omega^2 + \varepsilon\varphi(\tau)\big)\,\eta = 0, \tag{8}$$

where $(+)$ corresponds to the vertical equilibrium position of the pendulum $\theta = 0$ and $(-)$ to its inverted equilibrium $\theta = \pi$. According to Lyapunov's theorem on stability based on a linear approximation for a nonlinear system with periodic coefficients instability (asymptotic stability) of a solution is determined by instability (asymptotic stability) of the equations linearized about the solution. Hence, stability of solutions $\theta = 0$ or $\theta = \pi$ of (4) and (7) is equivalent to that of $\eta = 0$ of the equations in (8). According to the general stability theory by Floquet, see e.g. [7], a linear periodic system

$$\dot{x} = \mathbf{G}(\tau)x, \quad \mathbf{X}(0) = \mathbf{I}$$

is asymptotically stable if all *Floquet multipliers* $|\rho_j| < 1$, which are the eigenvalues of the *monodromy matrix* $F = \mathbf{X}(2\pi)$. In our problem we have $j = 1, 2$ and the state vector $(x_1, x_2)' = (\eta, \dot{\eta})'$ with the following matrices corresponding to systems in (8)

$$\mathbf{G}(\tau) = \begin{pmatrix} 0 & 1 \\ \mp\omega^2\,(1 + a\cos\tau) & -\beta\omega \end{pmatrix}, \qquad \mathbf{G}(\tau) = \begin{pmatrix} 0 & 1 \\ \mp\big(\omega^2 + \varepsilon\cos\tau\big) & -\beta\omega \end{pmatrix}. \tag{9}$$

Floquet multipliers for a two dimensional system can be found as follows

$$\begin{aligned} \rho_1 &= \frac{\operatorname{tr} F + \sqrt{(\operatorname{tr} F)^2 - 4\det F}}{2}, \\ \rho_2 &= \frac{\operatorname{tr} F - \sqrt{(\operatorname{tr} F)^2 - 4\det F}}{2}. \end{aligned}$$

Condition ($|\rho_1| < 1$ and $|\rho_2| < 1$) of asymptotic stability can be written in the following two cases as

(i) If $(\operatorname{tr} F)^2 \geq 4\det F$, then the condition of asymptotic stability writes as $|\operatorname{tr} F| - 1 < \det F < 1$.

(ii) If $(\operatorname{tr} F)^2 < 4\det F$, then the condition of asymptotic stability writes as $\det F < 1$.

Notice, that these two cases correspond to static and dynamic forms of loosing stability: divergence (i) and flatter (ii), see Chapter 13 in [7]. By Liouville's theorem, see e.g. [8], the determinant of monodromy matrix can be found as follows

$$\det F = \exp\left(\int_0^{2\pi} \operatorname{tr}\mathbf{G}(\tau)\,\mathrm{d}\tau\right) = \exp(-2\pi\beta\omega).$$

So with positive coefficients β and ω stability can be lost only by violation condition $|\operatorname{tr} F| - 1 < \det F$ in case (i). These conditions are checked numerically and the stability regions are depicted in Fig. 3 for different values of damping. We see that vertical position $\theta = 0$ primarily is stabilized by damping, i.e. stability domains basically expand with the increase of damping coefficients.

Stability of inverted position

As we see in Fig. 3 (for $q < 0$ see also Fig. 7) the inverted vertical position, $\theta = \pi$, see Figs. 5 and 6, can be not only stabilized but also destabilized by damping. Particularly the increase in damping coefficient shifts up (in excitation amplitude) the stability region of inverted position. In Figs. 3 and 7 regions of damping stabilization are denoted with blue color while regions of damping destabilization are depicted with red color.

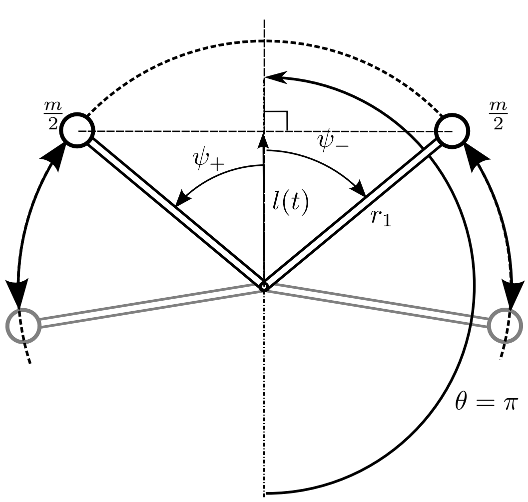

Figure 5: Inverted pendulum with shifting center of mass

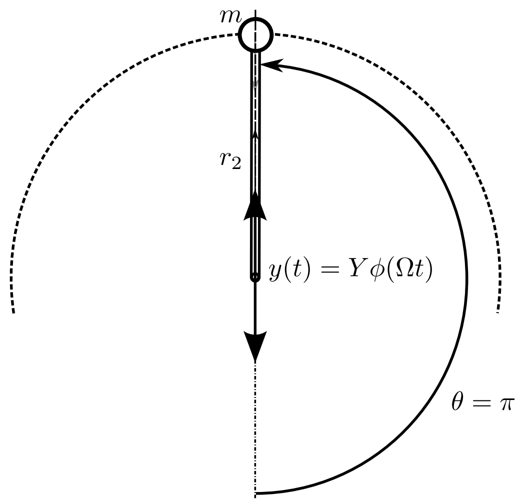

Figure 6: Inverted pendulum with vibrating support

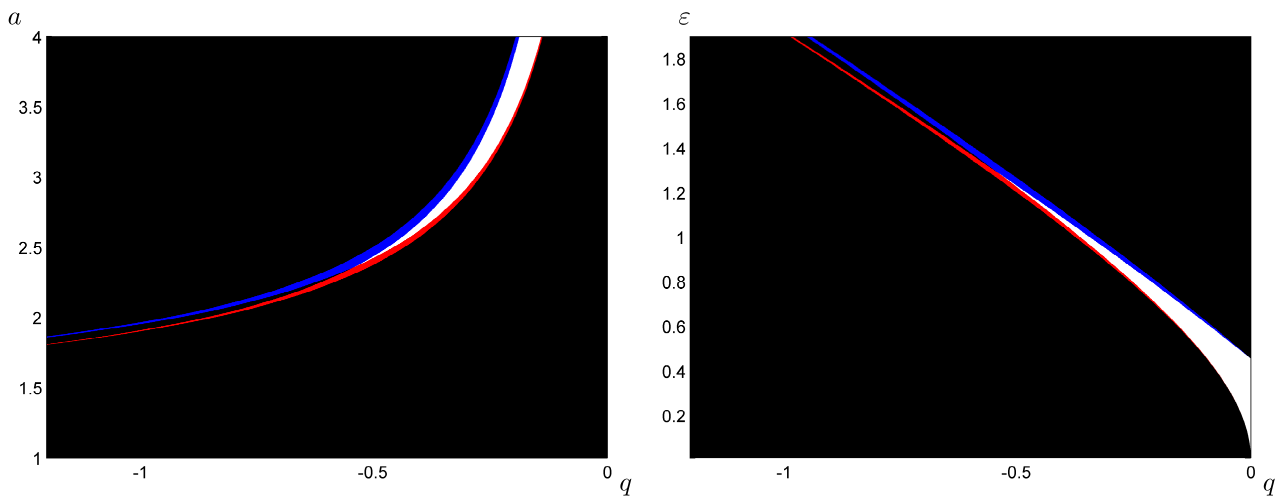

Figure 7: Ince-Strutt diagram for $\varphi(\tau) = \cos(\tau)$ depicts the stability of the inverted vertical pendulum position, $\theta = \pi$, depending on relative eigenfrequency $\omega = \sqrt{|q|}$ and relative excitation amplitude a (left) or ε (right), defined in (3) and (6). Left and right diagrams are the same up to the transformation $\varepsilon = a\omega^2$. Black color depicts the resonance domains for both $\beta = 0.0001$ and $\beta = 0.4$. Blue color marks the domains of damping stabilization with $\beta = 0.4$. While red denotes the domain where damping $\beta = 0.4$ destabilizes the vertical position ($\theta = \pi$).

Stability with stepwise excitation

If we choose piecewise constant periodic excitation $\varphi(\tau) = \frac{\pi}{4}\,\mathrm{sign}(\cos(\tau))$ drawn with solid line in Fig. 8 (top), such that its fist Fourier term is $\cos(\tau)$ drawn with dashed line in Fig. 8 (top), then we obtain very similar stability regions to those for $\varphi(\tau) = \cos(\tau)$. It is seen in Figs. 3 and 4 that stability regions differs only for big values of parameters ω and a (left) or ε (bottom). It is known, see [7], that non-degenerate regions of parametric resonance (i.e. instability) are approximated for small excitation amplitude and damping with the use of only corresponding harmonics of the excitation function. Monodromy matrix for piecewise constant excitation can be calculated analytically, similar to §28, Chapter 3 in [4], as multiplication of two matrix exponents $F = \exp(G(\pi))\exp(G(0))$, where matrices of systems are the following

$$\mathbf{G}(\tau) = \begin{pmatrix} 0 & 1 \\ \mp\omega^2\left(1 + a\frac{\pi}{4}\,\mathrm{sign}(\cos(\tau))\right) & -\beta\omega \end{pmatrix}, \qquad \mathbf{G}(\tau) = \begin{pmatrix} 0 & 1 \\ \mp\left(\omega^2 + \varepsilon\frac{\pi}{4}\,\mathrm{sign}(\cos(\tau))\right) & -\beta\omega \end{pmatrix}. \tag{10}$$

Notice that the pivot of the pendulum in Fig. 2 oscillates similarly, see Fig. 8 (right) for both excitation functions $\varphi(\tau)$ in Fig. 8 (left).

Conclusions

A new mechanical model of parametrically excited pendulum is proposed. Mathematically the model is equivalent to the pendulum with vertically vibrating pivot. For the new pendulum the stabilization of inverse vertical position is possible only when its center of mass periodically moves below the pivot, $l_a > l_0$. It is observed that the increase in damping coefficient shifts up (in excitation amplitude) the stability region of inverted position, thus stabilizing it for some parameters and destabilizing for others. Ince-Strutt diagrams for piecewise-constant periodic excitation and for excitation by its

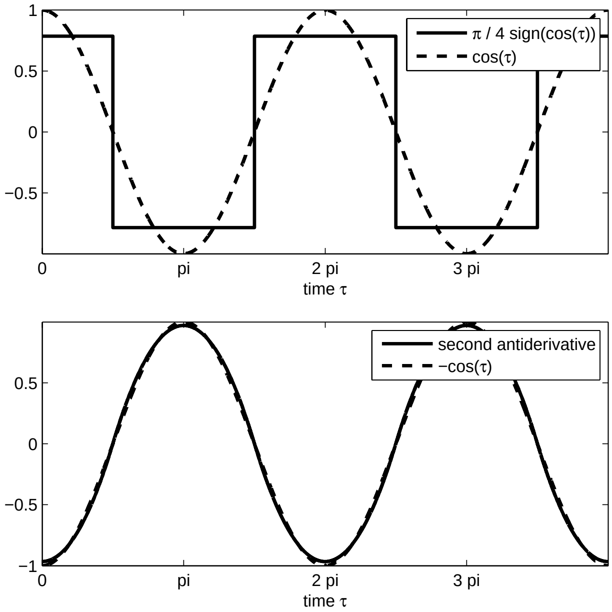

Figure 8: Stepwise excitation function $\varphi(\tau) = \frac{\pi}{4}\,\mathrm{sign}(\cos(\tau))$ (solid line, top) compared with excitation by its first harmonic $\varphi(\tau) = \cos(\tau)$ (dashed line, top) and comparison of their corresponding zero-mean second anti-derivatives $\phi(\tau)$ (bottom).

first harmonic appear to be very similar in the region, where parameters of relative eigenfrequency, ω, damping, β, and excitation amplitude, ε, have small values.

Acknowledgements
This research was supported by the Austrian Science Fund (FWF) under Grant P25979-N25.

References

[1] A. P. Seyranian, H. Yabuno, and K. Tsumoto. Instability and periodic motion of a physical pendulum with a vibrating suspension point (theoretical and experimental approach). *Doklady physics*, 50(9):467–472, 2005.

[2] A. A. Seyranian and A. P. Seyranian. The stability of an inverted pendulum with a vibrating suspension point. *Journal of applied mathematics and mechanics*, 70(5):754–761, 2006.

[3] B.S Bardin and A.P Markeyev. The stability of the equilibrium of a pendulum for vertical oscillations of the point of suspension. *Journal of Applied Mathematics and Mechanics*, 59(6):879–886, 1995.

[4] V. I. Arnold. *Ordinary Differential Equations*. Springer, New York, 1992.

[5] D. Yurchenko and P. Alevras. Dynamics of the n-pendulum and its application to a wave energy converter concept. *International Journal of Dynamics and Control*, pages 1–10, 2013.

[6] A. O. Belyakov, A. P. Seyranian, and A. Luongo. Dynamics of the pendulum with periodically varying length. *Physica D: Nonlinear Phenomena*, 238(16):1589–1597, 2009.

[7] A. P. Seyranian and A. A. Mailybaev. *Multiparameter Stability Theory with Mechanical Applications*. World Scientific, New Jersey, 2003.

[8] V. A. Yakubovich and V. M. Starzhinskii. *Parametric Resonance in Linear Systems*. Nauka, Moscow, 1987.