Another mechanical model of parametrically excited pendulum and stabilization of its inverted equilibrium position

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Scheme of a parametric pendulum



Equations of motion

$$mr_1^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c_1 \frac{\mathrm{d}\theta}{\mathrm{d}t} + mg \,l(t)\sin(\theta) = 0 \tag{1}$$

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Dimensionless equation

Three dimensionless parameters and new time $au = \Omega t$

$$a = \frac{l_a}{l_0}, \quad \omega = \frac{\Omega_0}{\Omega}, \quad \beta = \frac{c_1}{\Omega_0 m r_1^2},$$
 (2)

where

$$l(t) = l_0 + l_y \varphi(\Omega t) \leq r_1, \quad \Omega_0 = \frac{\sqrt{l_0 g}}{r_1}.$$

Dimensionless equation

$$\ddot{\theta} + \beta \omega \dot{\theta} + \omega^2 \left(1 + a\varphi(\tau) \right) \sin(\theta) = 0, \tag{3}$$

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the upper dot denotes differentiation with respect to new time τ .

Resonances

Function φ is a zero mean 2π -periodic excitation function, $\varphi(\tau + 2\pi) = \varphi(\tau)$.

Resonance relative frequencies

$$\omega_k=\frac{k}{2}, \quad k=1,2,\ldots.$$

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Resonance domains for $\varphi(\tau) = \cos(\tau)$



Ince-Strutt diagram depending on relative eigenfrequency $\omega = \sqrt{|q|}$ and relative excitation amplitude *a*. Black color depicts the resonance domains for both $\beta = 0.0001$ and $\beta = 0.4$. Blue color marks the domains of damping stabilization with $\beta = 0.4$. Red denotes the domain where damping $\beta = 0.4$ destabilizes the vertical position.

Stability of inverted pendulum



Linearized equation around both $\theta = 0$ and $\theta = \pi$

$$\ddot{\eta} + \beta \sqrt{|q|} \dot{\eta} + q \left(1 + a\varphi(\tau)\right) \eta = 0, \quad \sqrt{|q|} = \omega. \tag{4}$$

Schemes of equivalent parametric pendula



Equations of motion

$$mr_1^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c_1 \frac{\mathrm{d}\theta}{\mathrm{d}t} + mgl(t)\sin(\theta) = 0 \quad , \quad (5)$$
$$mr_2^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c_2 \frac{\mathrm{d}\theta}{\mathrm{d}t} + mr_2 \left(g - \frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}\right)\sin(\theta) = 0 \quad . \quad (6)$$

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Conditions for the same equation of motion

$$\varphi(\tau) = -\ddot{\phi}(\tau), \quad l_0 = \frac{r_1^2}{r_2}, \quad l_y = \frac{r_1^2 \Omega^2 Y}{r_2 g}, \quad \frac{c_1}{r_1^2} = \frac{c_2}{r_2^2}.$$
 (7)

where $l(t) = l_0 + l_y \varphi(\Omega t) \le r_1$, implying the same eigenfrequency

$$\Omega_0 = \frac{\sqrt{l_0 g}}{r_1} = \sqrt{\frac{g}{r_2}}$$

Three dimensionless parameters and new time $au = \Omega t$

$$\varepsilon = a \omega^2 = \frac{\Omega_0^2 Y}{g}, \quad \omega = \frac{\Omega_0}{\Omega}, \quad \beta = \frac{c_1}{\Omega_0 m r_1^2} = \frac{c_2}{\Omega_0 m r_2^2}.$$
 (8)

Dimensionless equation

$$\ddot{\theta} + \beta \omega \dot{\theta} + \left(\omega^2 + \varepsilon \varphi(\tau) \right) \sin(\theta) = 0, \tag{9}$$

the upper dot denotes differentiation with respect to new time τ .

Resonance domains for $\varphi(\tau) = \cos(\tau)$, related $\varepsilon = a \omega^2$





Stability of inverted position for $\varphi(\tau) = \cos(\tau)$



Figure: Ince-Strutt diagram for $\varphi(\tau) = \cos(\tau)$ depicts the stability of the inverted vertical pendulum position, $\theta = \pi$, depending on relative eigenfrequency $\omega = \sqrt{|q|}$ and relative excitation amplitude *a* (left) or ε (right). Left and right diagrams are the same up to the transformation $\varepsilon = a\omega^2$. Black color depicts the resonance domains for both $\beta = 0.0001$ and $\beta = 0.4$. Blue color marks the domains of damping stabilization with $\beta = 0.4$. While red denotes the domain where damping $\beta = 0.4$ destabilizes the vertical position ($\theta = \pi$).

Stability analysis

Let us perturbed solutions of equations

$$\begin{split} \ddot{\theta} + \beta \omega \dot{\theta} + \omega^2 \left(1 + a \varphi(\tau) \right) \sin(\theta) &= 0, \\ \ddot{\theta} + \beta \omega \dot{\theta} + \left(\omega^2 + \varepsilon \varphi(\tau) \right) \sin(\theta) &= 0 \end{split}$$

by small η , then we have the corresponding linearized equations:

$$\begin{split} \ddot{\eta} + \beta \omega \dot{\eta} \pm \omega^2 \left(1 + \mathbf{a} \varphi(\tau) \right) \eta &= 0, \\ \ddot{\eta} + \beta \omega \dot{\eta} \pm \left(\omega^2 + \varepsilon \varphi(\tau) \right) \eta &= 0, \end{split}$$

where (+) corresponds to the vertical equilibrium position of the pendulum $\theta = 0$ and (-) to its inverted equilibrium $\theta = \pi$.

Linear periodic system, $(x_1, x_2)' = (\eta, \dot{\eta})'$

$$\dot{x} = \mathbf{G}(\tau)x, \quad \mathbf{X}(0) = \mathbf{I}$$

Stability of a linear periodic system $\dot{x} = \mathbf{G}(\tau)x, \ \mathbf{X}(0) = \mathbf{I}$

Asymptotically stable if all *Floquet multipliers* $|\rho_j| < 1$, which are the eigenvalues of the *monodromy matrix* $F = \mathbf{X}(2\pi)$.

Floquet multipliers can be found for dim x = 2 as follows

$$\rho_1 = \frac{\operatorname{tr} F + \sqrt{(\operatorname{tr} F)^2 - 4 \det F}}{2}, \quad \rho_2 = \frac{\operatorname{tr} F - \sqrt{(\operatorname{tr} F)^2 - 4 \det F}}{2}$$

Condition $(|
ho_1| < 1$ and $|
ho_2| < 1)$ of asymptotic stability

(i) If
$$(tr F)^2 \ge 4 \det F$$
, then $|tr F| < 1 + \det F < 1$.
(ii) If $(tr F)^2 < 4 \det F$, then $\det F < 1$.

These are static and dynamic forms of loosing stability: divergence (i) and flatter (ii). By Liouville's theorem det F < 1:

$$\det F = \exp\left(\int_0^{2\pi} \operatorname{tr} \mathbf{G}(\tau) \, \mathrm{d}\tau\right) = \exp(-2\pi\beta\omega),$$

Stability of a linear periodic system $\dot{x} = \mathbf{G}(\tau)x, \ \mathbf{X}(0) = \mathbf{I}$

Pendulum with vibrating mass center

$$\mathbf{G}(\tau) = \begin{pmatrix} 0 & 1 \\ \mp \omega^2 \left(1 + \mathbf{a} \varphi(\tau)\right) & -\beta \omega \end{pmatrix}.$$

Pendulum with vibrating pivot

$$old G(au) \;\;=\;\; egin{pmatrix} 0 & 1 \ \mp \left(\omega^2 + arepsilon arphi(au)
ight) & -eta \omega \end{pmatrix}.$$

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Stepwise excitation



Stepwise excitation function $\varphi(\tau) = \frac{\pi}{4} \operatorname{sign}(\cos(\tau))$ (solid line, top) compared with excitation by its first harmonic $\varphi(\tau) = \cos(\tau)$ (dashed line, top) and comparison of their corresponding zero-mean second anti-derivatives $\phi(\tau)$ (bottom).

Stability analysis with stepwise excitation

Monodromy matrix for piecewise constant excitation can be calculated analytically as multiplication of two matrix exponents

 $F = \exp(G(\pi)) \exp(G(0)),$

where matrices of the linearized systems are the following:

Pendulum with vibrating mass center

$$\mathbf{G}(0) = egin{pmatrix} 0 & 1 \ \mp \omega^2 \left(1 + a rac{\pi}{4}
ight) & -eta \omega \end{pmatrix}, \quad \mathbf{G}(\pi) = egin{pmatrix} 0 & 1 \ \mp \omega^2 \left(1 - a rac{\pi}{4}
ight) & -eta \omega \end{pmatrix},$$

Pendulum with vibrating pivot

$$\mathbf{G}(0) = \begin{pmatrix} 0 & 1 \\ \mp \left(\omega^2 + \varepsilon \frac{\pi}{4}\right) & -\beta\omega \end{pmatrix}, \quad \mathbf{G}(\pi) = \begin{pmatrix} 0 & 1 \\ \mp \left(\omega^2 - \varepsilon \frac{\pi}{4}\right) & -\beta\omega \end{pmatrix}.$$

Comparison of resonance domains



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Conclusions

- 1. A new mechanical model of parametrically excited pendulum is proposed.
- 2. Mathematically the model is equivalent to the pendulum with vertically vibrating pivot.
- 3. For the new pendulum the stabilization of inverse vertical position is possible only when its center of mass periodically moves below the pivot, $l_y > l_0$.
- 4. There are both frequency stabilization and destabilization of inverse vertical position
- 5. For small damping and excitation amplitude, instability domains are very similar for excitations with the same first harmonics.