Another mechanical model of parametrically excited pendulum and stabilization of its inverted equilibrium position

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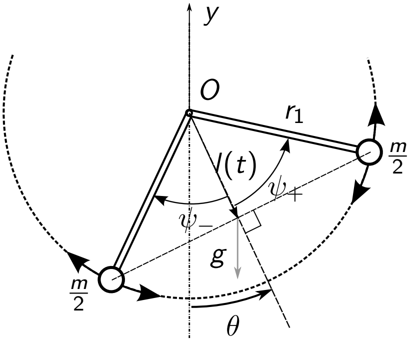
Scheme of a parametric pendulum



Equations of motion

$$mr_1^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c_1 \frac{\mathrm{d}\theta}{\mathrm{d}t} + m\, g\, l(t) \sin(\theta) = 0 \tag{1}$$

Dimensionless equation

Three dimensionless parameters and new time $\tau = \Omega t$

$$a = \frac{l_a}{l_0}, \quad \omega = \frac{\Omega_0}{\Omega}, \quad \beta = \frac{c_1}{\Omega_0 m r_1^2}, \tag{2}$$

where

$$l(t) = l_0 + l_y \varphi(\Omega t) \leq r_1, \quad \Omega_0 = \frac{\sqrt{l_0\, g}}{r_1}.$$

Dimensionless equation

$$\ddot{\theta} + \beta\omega\dot{\theta} + \omega^2 \left(1 + a\varphi(\tau)\right) \sin(\theta) = 0, \tag{3}$$

the upper dot denotes differentiation with respect to new time τ.

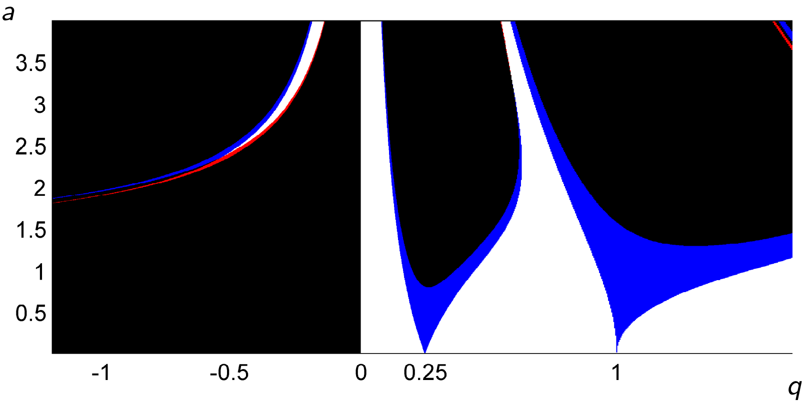
Resonances

Function φ is a zero mean 2π-periodic excitation function, $\varphi(\tau + 2\pi) = \varphi(\tau)$.

Resonance relative frequencies

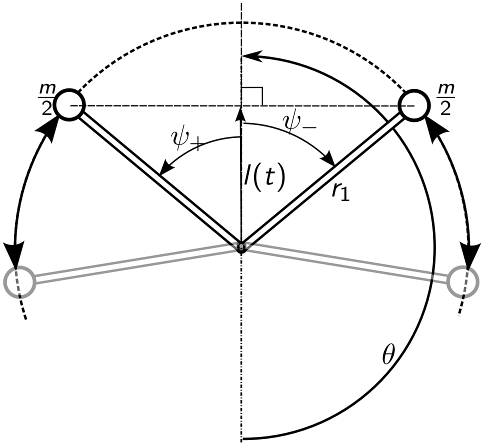
$$\omega_k = \frac{k}{2}, \quad k = 1, 2, \ldots .$$

Resonance domains for $\varphi(\tau) = \cos(\tau)$



Ince-Strutt diagram depending on relative eigenfrequency $\omega = \sqrt{|q|}$ and relative excitation amplitude *a*. Black color depicts the resonance domains for both $\beta = 0.0001$ and $\beta = 0.4$. Blue color marks the domains of damping stabilization with $\beta = 0.4$. Red denotes the domain where damping $\beta = 0.4$ destabilizes the vertical position.

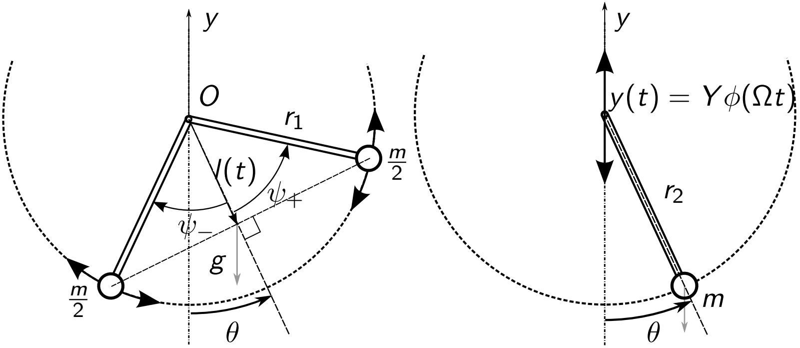
Stability of inverted pendulum



Linearized equation around both $\theta = 0$ and $\theta = \pi$

$$\ddot{\eta} + \beta\sqrt{|q|}\dot{\eta} + q\left(1 + a\varphi(\tau)\right)\eta = 0, \quad \sqrt{|q|} = \omega. \tag{4}$$

Schemes of equivalent parametric pendula



Equations of motion

$$m r_1^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c_1 \frac{\mathrm{d}\theta}{\mathrm{d}t} + m\, g\, l(t) \sin(\theta) = 0 \quad , \qquad (5)$$

$$m r_2^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c_2 \frac{\mathrm{d}\theta}{\mathrm{d}t} + m\, r_2 \left(g - \frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} \right) \sin(\theta) = 0 \quad . \qquad (6)$$

Conditions for the same equation of motion

$$\varphi(\tau) = -\ddot{\phi}(\tau), \quad l_0 = \frac{r_1^2}{r_2}, \quad l_y = \frac{r_1^2 \Omega^2 Y}{r_2 g}, \quad \frac{c_1}{r_1^2} = \frac{c_2}{r_2^2}. \tag{7}$$

where $l(t) = l_0 + l_y \varphi(\Omega t) \leq r_1$, implying the same eigenfrequency

$$\Omega_0 = \frac{\sqrt{l_0\, g}}{r_1} = \sqrt{\frac{g}{r_2}}.$$

Three dimensionless parameters and new time $\tau = \Omega t$

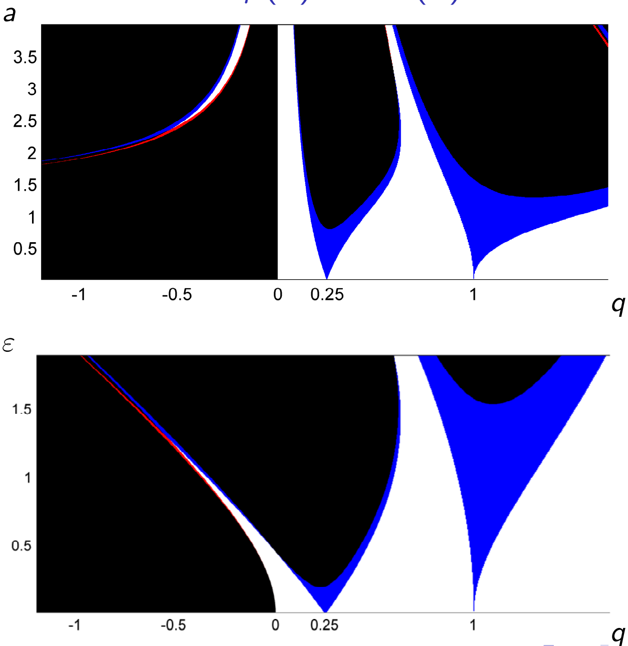
$$\varepsilon = a\,\omega^2 = \frac{\Omega_0^2 Y}{g}, \quad \omega = \frac{\Omega_0}{\Omega}, \quad \beta = \frac{c_1}{\Omega_0 m r_1^2} = \frac{c_2}{\Omega_0 m r_2^2}. \tag{8}$$

Dimensionless equation

$$\ddot{\theta} + \beta\omega\dot{\theta} + \left(\omega^2 + \varepsilon\varphi(\tau)\right)\sin(\theta) = 0, \tag{9}$$

the upper dot denotes differentiation with respect to new time τ.

Resonance domains for $\varphi(\tau) = \cos(\tau)$, related $\varepsilon = a\omega^2$



Stability of inverted position for $\varphi(\tau) = \cos(\tau)$

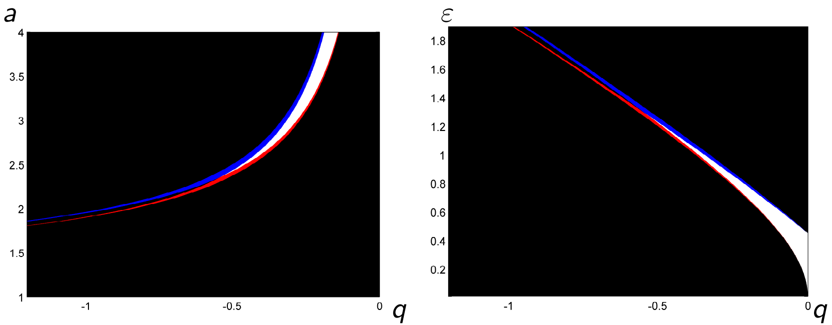


Figure: Ince-Strutt diagram for $\varphi(\tau) = \cos(\tau)$ depicts the stability of the inverted vertical pendulum position, $\theta = \pi$, depending on relative eigenfrequency $\omega = \sqrt{|q|}$ and relative excitation amplitude a (left) or ε (right). Left and right diagrams are the same up to the transformation $\varepsilon = a\omega^2$. Black color depicts the resonance domains for both $\beta = 0.0001$ and $\beta = 0.4$. Blue color marks the domains of damping stabilization with $\beta = 0.4$. While red denotes the domain where damping $\beta = 0.4$ destabilizes the vertical position ($\theta = \pi$).

Stability analysis

Let us perturbed solutions of equations

$$\begin{aligned} \ddot{\theta} + \beta\omega\dot{\theta} + \omega^2 \left(1 + a\varphi(\tau)\right) \sin(\theta) &= 0, \\ \ddot{\theta} + \beta\omega\dot{\theta} + \left(\omega^2 + \varepsilon\varphi(\tau)\right) \sin(\theta) &= 0 \end{aligned}$$

by small η, then we have the corresponding linearized equations:

$$\begin{aligned} \ddot{\eta} + \beta\omega\dot{\eta} \pm \omega^2 \left(1 + a\varphi(\tau)\right) \eta &= 0, \\ \ddot{\eta} + \beta\omega\dot{\eta} \pm \left(\omega^2 + \varepsilon\varphi(\tau)\right) \eta &= 0, \end{aligned}$$

where $(+)$ corresponds to the vertical equilibrium position of the pendulum $\theta = 0$ and $(-)$ to its inverted equilibrium $\theta = \pi$.

Linear periodic system, $(x_1, x_2)' = (\eta, \dot{\eta})'$

$$\dot{x} = \mathbf{G}(\tau)x, \quad \mathbf{X}(0) = \mathbf{I}$$

Stability of a linear periodic system $\dot{x} = \mathbf{G}(\tau)x$, $\mathbf{X}(0) = \mathbf{I}$

Asymptotically stable if all *Floquet multipliers* $|\rho_j| < 1$, which are the eigenvalues of the *monodromy matrix* $F = \mathbf{X}(2\pi)$.

Floquet multipliers can be found for $\dim x = 2$ as follows

$$\rho_1 = \frac{\operatorname{tr} F + \sqrt{(\operatorname{tr} F)^2 - 4\det F}}{2}, \quad \rho_2 = \frac{\operatorname{tr} F - \sqrt{(\operatorname{tr} F)^2 - 4\det F}}{2}.$$

Condition ($|\rho_1| < 1$ and $|\rho_2| < 1$) of asymptotic stability

(i) If $(\operatorname{tr} F)^2 \geq 4\det F$, then $|\operatorname{tr} F| < 1 + \det F < 1$.

(ii) If $(\operatorname{tr} F)^2 < 4\det F$, then $\det F < 1$.

These are static and dynamic forms of loosing stability: divergence (i) and flatter (ii). By Liouville's theorem $\det F < 1$:

$$\det F = \exp\left(\int_0^{2\pi} \operatorname{tr} \mathbf{G}(\tau)\, \mathrm{d}\tau\right) = \exp(-2\pi\beta\omega),$$

Stability of a linear periodic system $\dot{x} = \mathbf{G}(\tau)x$, $\mathbf{X}(0) = \mathbf{I}$

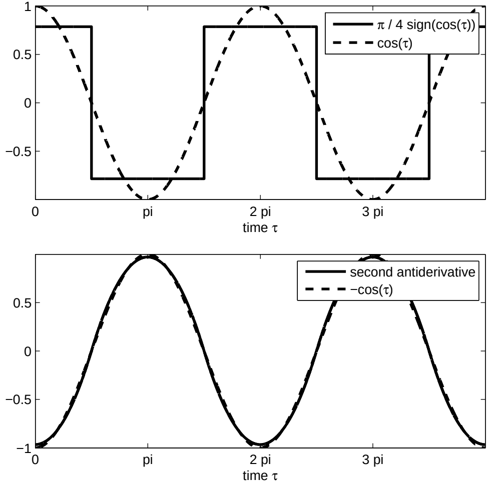
Pendulum with vibrating mass center

$$\mathbf{G}(\tau) = \begin{pmatrix} 0 & 1 \\ \mp\omega^2 \left(1 + a\varphi(\tau)\right) & -\beta\omega \end{pmatrix}.$$

Pendulum with vibrating pivot

$$\mathbf{G}(\tau) = \begin{pmatrix} 0 & 1 \\ \mp \left(\omega^2 + \varepsilon\varphi(\tau)\right) & -\beta\omega \end{pmatrix}.$$

Stepwise excitation



Stepwise excitation function $\varphi(\tau) = \frac{\pi}{4}\,\text{sign}(\cos(\tau))$ (solid line, top) compared with excitation by its first harmonic $\varphi(\tau) = \cos(\tau)$ (dashed line, top) and comparison of their corresponding zero-mean second anti-derivatives $\phi(\tau)$ (bottom).

Stability analysis with stepwise excitation

Monodromy matrix for piecewise constant excitation can be calculated analytically as multiplication of two matrix exponents

$$F = \exp(G(\pi))\exp(G(0)),$$

where matrices of the linearized systems are the following:

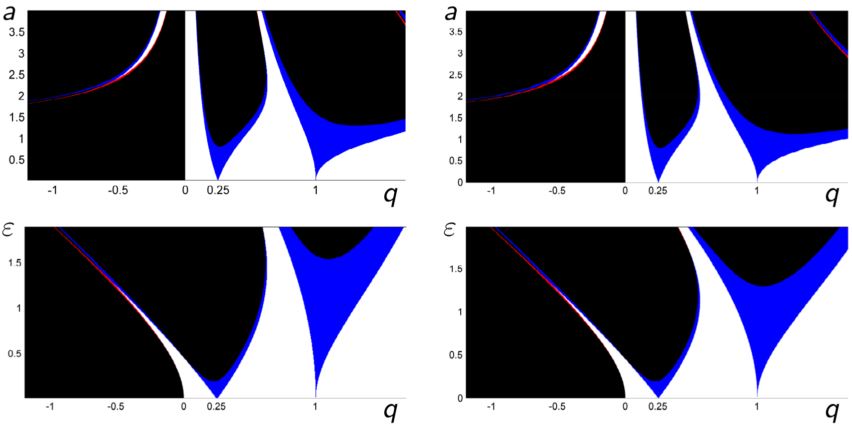
Pendulum with vibrating mass center

$$\mathbf{G}(0) = \begin{pmatrix} 0 & 1 \\ \mp\omega^2\left(1 + a\frac{\pi}{4}\right) & -\beta\omega \end{pmatrix}, \quad \mathbf{G}(\pi) = \begin{pmatrix} 0 & 1 \\ \mp\omega^2\left(1 - a\frac{\pi}{4}\right) & -\beta\omega \end{pmatrix}.$$

Pendulum with vibrating pivot

$$\mathbf{G}(0) = \begin{pmatrix} 0 & 1 \\ \mp\left(\omega^2 + \varepsilon\frac{\pi}{4}\right) & -\beta\omega \end{pmatrix}, \quad \mathbf{G}(\pi) = \begin{pmatrix} 0 & 1 \\ \mp\left(\omega^2 - \varepsilon\frac{\pi}{4}\right) & -\beta\omega \end{pmatrix}.$$

Comparison of resonance domains



$\varphi(\tau) = \cos(\tau)$

$\varphi(\tau) = \frac{\pi}{4}\,\mathrm{sign}(\cos(\tau))$

Conclusions

1. A new mechanical model of parametrically excited pendulum is proposed.
2. Mathematically the model is equivalent to the pendulum with vertically vibrating pivot.
3. For the new pendulum the stabilization of inverse vertical position is possible only when its center of mass periodically moves below the pivot, $l_y > l_0$.
4. There are both frequency stabilization and destabilization of inverse vertical position
5. For small damping and excitation amplitude, instability domains are very similar for excitations with the same first harmonics.