Twirling of Hula-hoop: New Results

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Literature

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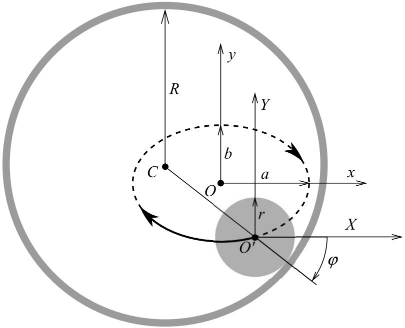
Formulation of the problem



$x = a \sin \omega t, \quad y = b \cos \omega t$

θ – angle about the mass center C

$I_C = mR^2$ – central moment of inertia

m and R – mass and radius of the hula-hoop

k – coefficient of viscous friction

r – radius of the waist

F_T – friction force

N – reaction force

φ – angle between x and CO'

$$
\begin{aligned}
&I_C \ddot{\theta} + k\dot{\theta} = -F_T R \\
&m\,(R - r)\,\ddot{\varphi} = m\,(\ddot{x} \sin \varphi + \ddot{y} \cos \varphi) + F_T \\
&m\,(R - r)\,\dot{\varphi}^2 = m\,(\ddot{x} \cos \varphi - \ddot{y} \sin \varphi) + N
\end{aligned}
$$

$(R - r)\,\dot{\varphi} = R\dot{\theta}$ – non-slippage condition,

$N > 0$ – non-separability condition

Equation of motion and non-separability condition

$$\ddot{\varphi} + \frac{k}{2mR^2\omega}\dot{\varphi} + \frac{\omega^2\left(a\sin\omega t\sin\varphi + b\cos\omega t\cos\varphi\right)}{2\left(R-r\right)} = 0$$

$$N = m\left(R-r\right)\dot{\varphi}^2 + m\,\omega^2\left(a\sin\omega t\cos\varphi - b\cos\omega t\sin\varphi\right) > 0$$

we use simple trigonometric relations

$$a\sin\omega t\sin\varphi + b\cos\omega t\cos\varphi = \tfrac{a+b}{2}\cos(\omega t-\varphi) - \tfrac{a-b}{2}\cos(\omega t+\varphi),$$
$$a\sin\omega t\cos\varphi - b\cos\omega t\sin\varphi = \tfrac{a+b}{2}\sin(\omega t-\varphi) + \tfrac{a-b}{2}\sin(\omega t+\varphi),$$

we introduce new time $\tau = \omega t$ and dimensionless parameters

$$\gamma = \frac{k}{2mR^2\omega}, \quad \mu = \frac{a+b}{4(R-r)} > 0, \quad \varepsilon = \frac{a-b}{4(R-r)} \geq 0,$$

$$\ddot{\varphi} + \gamma\dot{\varphi} + \mu\cos(\varphi-\tau) = \varepsilon\cos(\varphi+\tau) \qquad (1)$$

$$\dot{\varphi}^2 - 2\mu\sin(\varphi-\tau) + 2\varepsilon\cos(\varphi+\tau) > 0, \qquad (2)$$

Exact solution of the unperturbed equation

The unperturbed equation ($\varepsilon = 0$)

$$\ddot{\varphi} + \gamma\dot{\varphi} + \mu\cos(\varphi - \tau) = 0 \qquad (3)$$

has the exact solutions $\varphi = \tau + \varphi_0$ if $|\gamma| \le \mu$,
where constants $\varphi_0 \mod 2\pi$ are defined from $\gamma + \mu\cos\varphi_0 = 0$.
Asymptotic stability conditions

$$\gamma > 0, \quad \mu\sin\varphi_0 < 0, \qquad (4)$$

yield $0 < \gamma < \mu$.
Inseparability condition (2) takes the form

$$1 - 2\mu\sin\varphi_0 > 0 \qquad (5)$$

The rotation is asymptotically stable and inseparable with $\varphi_0 = -\arccos(-\gamma/\mu)$, while unstable rotation with $\varphi_0 = \arccos(-\gamma/\mu)$ is inseparable only if $\mu < \sqrt{1/4 + \gamma^2}$.

Approximate solution ($\varepsilon \neq 0$)

We represent the solution as the series $\varphi = \tau + \varphi_0 + \varepsilon\varphi_1(\tau) + \dots$

$$\varepsilon^0 : \qquad \gamma + \mu \cos \varphi_0 = 0, \tag{6}$$

$$\varepsilon^1 : \qquad \ddot{\varphi}_1 + \gamma\dot{\varphi}_1 - \mu \sin(\varphi_0)\, \varphi_1 = \cos(\varphi_0 + 2\tau), \tag{7}$$

where we take $\varphi_0 = -\arccos(-\gamma/\mu)$ corresponding to the stable solution of the unperturbed system. Thus, (7) takes the form

$$\ddot{\varphi}_1 + \gamma\dot{\varphi}_1 + \sqrt{\mu^2 - \gamma^2}\, \varphi_1 = \cos(\varphi_0 + 2\tau) \tag{8}$$

and has the unique periodic solution

$$\varphi_1(\tau) \quad = \quad C \sin(\varphi_0 + 2\tau) + D \cos(\varphi_0 + 2\tau) \tag{9}$$

where $\quad C = \frac{2\gamma}{3\gamma^2+\mu^2-8\sqrt{\mu^2-\gamma^2}+16}, \quad D = \frac{-4+\sqrt{\mu^2-\gamma^2}}{3\gamma^2+\mu^2-8\sqrt{\mu^2-\gamma^2}+16}$

Stability analysis

We add a small perturbation $\varphi = \varphi_* + u$ to the true solution φ_* of (1) and linearize (1) w.r.t. u obtaining the Mathieu-Hill equation with damping

$$\ddot{u} + \gamma \dot{u} + \left(\sqrt{\mu^2 - \gamma^2} + \varepsilon \Phi(2\tau) \right) u = 0, \tag{10}$$

where $\Phi = (\gamma C + 1)\sin(2\tau + \varphi_0) + \gamma D \cos(2\tau + \varphi_0) + O(\varepsilon)$.
Stability condition (absence of parametric resonance in (10))

$$\varepsilon < \frac{2\gamma}{\sqrt{(\gamma C + 1)^2 + \gamma^2 D^2}} + o(\varepsilon) \tag{11}$$

is also the stability condition for the original equation (1) according to the Lyapunov's theorem.

Non-separability condition in the first approximation

Solution

$\varphi = \tau - \arccos\left(-\frac{\gamma}{\mu}\right) + \varepsilon C \sin(\varphi_0 + 2\tau) + \varepsilon D \cos(\varphi_0 + 2\tau) + o(\varepsilon)$

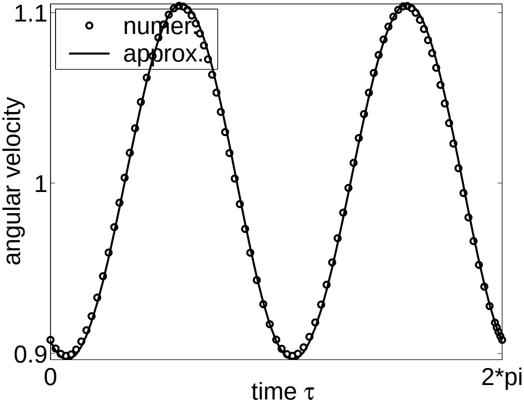
We substitute it into condition (2) that the hula-hoop is not separated from the waist

$$\dot{\varphi}^2 - 2\mu \sin(\varphi - \tau) + 2\varepsilon \cos(\varphi + \tau) > 0,$$

which is guaranteed for all τ if the following inequality is satisfied

$$\varepsilon < \frac{1 + 2\sqrt{\mu^2 - \gamma^2}}{2} \sqrt{\frac{\mu^2 + 3\gamma^2 - 8\sqrt{\mu^2 - \gamma^2} + 16}{\mu^2 + 8\gamma^2 - 12\sqrt{\mu^2 - \gamma^2} + 36}} + o(\varepsilon).$$

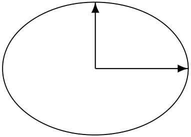
Comparison between approximate and numerical solutions



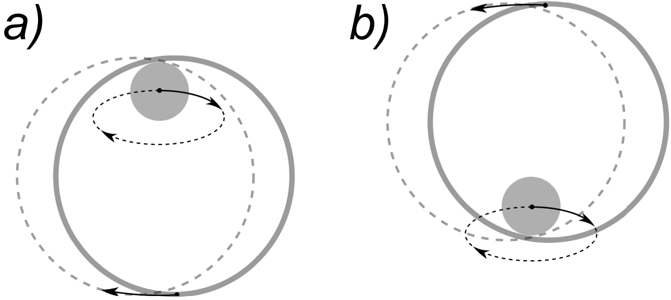
Angular velocity $\dot{\varphi}$
first approximation (solid line)
numerical simulation (circles).

$\varepsilon = 0.2$
$\mu = 1.2$
$\gamma = 1.0$

$$\frac{a}{2(R-r)} = 1.4$$
$$\frac{b}{2(R-r)} = 1.0$$



If μ and γ are also small of order ε then both direct and reverse rotation are possible



Approximate solution when μ and γ are small

We represent the solution as the series

$$\varphi = \rho\tau + \varphi_0 + \varepsilon\varphi_1(\tau) + \ldots \tag{12}$$

We introduce new not small parameters $\tilde{\mu} = \mu/\varepsilon, \quad \tilde{\gamma} = \gamma/\varepsilon$.
First adjustment φ_1 is defined by the equation:

$$\ddot{\varphi}_1 = \cos(\rho\tau + \varphi_0 + \tau) - \tilde{\mu}\cos(\rho\tau + \varphi_0 - \tau) - \tilde{\gamma}\rho \tag{13}$$

By equating separately constant and oscillation terms we find that solutions exist only if angular velocity $\rho \in \{1, 0, -1\}$.
With $\rho = 1$ we get the first order adjustment

$$\varphi_1(\tau) = -\frac{1}{4}\cos(\varphi_0 + 2\tau)\,, \quad \cos\varphi_0 = -\frac{\gamma}{\mu}$$

Stability and non-separability conditions with small μ и γ

Approximate solution with $\rho = 1$

$$\varphi = \tau + \varphi_0 - \frac{\varepsilon}{4}\cos(\varphi_0 + 2\tau) + o(\varepsilon), \quad \cos\varphi_0 = -\frac{\gamma}{\mu} \tag{14}$$

Stability conditions in first approximation

$$0 < \gamma < \mu, \quad \sin\varphi_0 < 0 \Rightarrow \varphi_0 = -\arccos\left(-\frac{\gamma}{\mu}\right) \mod 2\pi$$

Condition that the hula-hoop is not separated from the waist

$$\varepsilon < \frac{1 + 2\sqrt{\mu^2 - \gamma^2}}{3} + o(\varepsilon). \tag{15}$$

Stability and non-separability conditions with small μ и γ

Approximate solution with $\rho = -1$

$$\varphi = -\tau + \varphi_0 + \frac{\mu}{4}\cos(\varphi_0 - 2\tau) + o(\varepsilon), \quad \cos\varphi_0 = -\frac{\gamma}{\varepsilon} \tag{16}$$

Stability conditions in first approximation

$$0 < \gamma < \varepsilon, \quad \sin\varphi_0 > 0 \Rightarrow \varphi_0 = \arccos\left(-\frac{\gamma}{\varepsilon}\right) \mod 2\pi$$

Condition that the hula-hoop is not separated from the waist

$$\mu < \frac{1 + 2\sqrt{\varepsilon^2 - \gamma^2}}{3} + o(\varepsilon). \tag{17}$$

Condition of coexistence of direct and reverse rotations

Condition of coexistence direct and reverse rotations

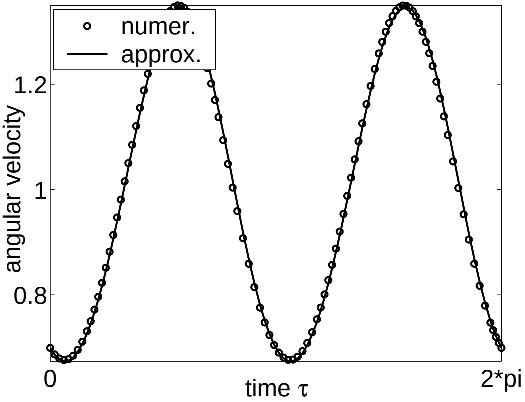
$$0 < \gamma \ < \min\{\varepsilon, \mu\} \tag{18}$$

is obtained from combination of $0 < \gamma < \mu$ (for direct rotation) and $0 < \gamma < \varepsilon$ (for reverse rotation), where both parameters ε and μ are supposed to be small.
In physical variables (18) takes the form

$$0 < 2k \frac{R - r}{R^2 \omega m} < a - |b| \tag{19}$$

i.e. the trajectory of the waist center should be sufficiently prolate.

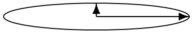
Angular velocity of direct rotation



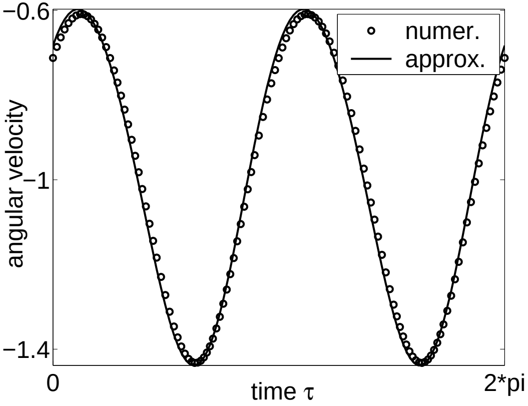
Angular velocity $\dot{\varphi}$
first approximation (solid line)
numerical simulation (circles).

$\varepsilon = 0.6$
$\mu = 0.8$
$\gamma = 0.5$

$$\frac{a}{2(R-r)} = 1.4$$
$$\frac{b}{2(R-r)} = 0.2$$



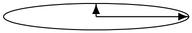
Angular velocity of reverse rotation



Angular velocity $\dot{\varphi}$
first approximation (solid line)
numerical simulation (circles).

$\varepsilon = 0.6$
$\mu = 0.8$
$\gamma = 0.5$

$$\frac{a}{2(R-r)} = 1.4$$
$$\frac{b}{2(R-r)} = 0.2$$



Conclusion

- Exact solutions for the hula-hoop under a circular excitation are obtained and their stability is studied
- Approximate solutions for an elliptic excitation are found
- The non-separability condition of the hula-hoop from the waist of a gymnast during rotation is derived
- The coexisting rotations for the direct and reverse rotations of the hula-hoop are analyzed
- The analytical solutions are compared with the results of numerical simulation

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