



## Twirling of Hula-hoop: New Results

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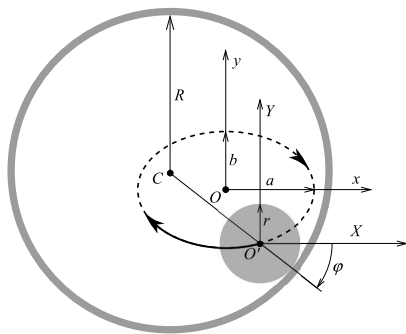
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# Literature

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## Formulation of the problem



$$x = a \sin \omega t, \quad y = b \cos \omega t$$

$\theta$  – angle about the mass center  $C$   
 $I_C = mR^2$  – central moment of inertia  
 $m$  and  $R$  – mass and radius of the hula-hoop  
 $k$  – coefficient of viscous friction  
 $r$  – radius of the waist  
 $F_T$  – friction force  
 $N$  – reaction force  
 $\varphi$  – angle between  $x$  and  $CO'$

$$I_C \ddot{\theta} + k\dot{\theta} = -F_T R$$

$$m(R - r) \ddot{\varphi} = m(\ddot{x} \sin \varphi + \ddot{y} \cos \varphi) + F_T$$

$$m(R - r) \dot{\varphi}^2 = m(\ddot{x} \cos \varphi - \ddot{y} \sin \varphi) + N$$

$(R - r) \dot{\varphi} = R\dot{\theta}$  – non-slippage condition,

$N > 0$  – non-separability condition

## Equation of motion and non-separability condition

$$\ddot{\varphi} + \frac{k}{2mR^2\omega} \dot{\varphi} + \frac{\omega^2 (a \sin \omega t \sin \varphi + b \cos \omega t \cos \varphi)}{2(R-r)} = 0$$

$$N = m(R-r)\dot{\varphi}^2 + m\omega^2 (a \sin \omega t \cos \varphi - b \cos \omega t \sin \varphi) > 0$$

we use simple trigonometric relations

$$\begin{aligned} a \sin \omega t \sin \varphi + b \cos \omega t \cos \varphi &= \frac{a+b}{2} \cos(\omega t - \varphi) - \frac{a-b}{2} \cos(\omega t + \varphi), \\ a \sin \omega t \cos \varphi - b \cos \omega t \sin \varphi &= \frac{a+b}{2} \sin(\omega t - \varphi) + \frac{a-b}{2} \sin(\omega t + \varphi), \end{aligned}$$

we introduce new time  $\tau = \omega t$  and dimensionless parameters

$$\gamma = \frac{k}{2mR^2\omega}, \quad \mu = \frac{a+b}{4(R-r)} > 0, \quad \varepsilon = \frac{a-b}{4(R-r)} \geq 0,$$

$$\ddot{\varphi} + \gamma \dot{\varphi} + \mu \cos(\varphi - \tau) = \varepsilon \cos(\varphi + \tau) \quad (1)$$

$$\dot{\varphi}^2 - 2\mu \sin(\varphi - \tau) + 2\varepsilon \cos(\varphi + \tau) > 0, \quad (2)$$

## Exact solution of the unperturbed equation

The unperturbed equation ( $\varepsilon = 0$ )

$$\ddot{\varphi} + \gamma \dot{\varphi} + \mu \cos(\varphi - \tau) = 0 \quad (3)$$

has the exact solutions  $\varphi = \tau + \varphi_0$  if  $|\gamma| \leq \mu$ , where constants  $\varphi_0 \bmod 2\pi$  are defined from  $\gamma + \mu \cos \varphi_0 = 0$ . Asymptotic stability conditions

$$\gamma > 0, \quad \mu \sin \varphi_0 < 0, \quad (4)$$

yield  $0 < \gamma < \mu$ .

Inseparability condition (2) takes the form

$$1 - 2\mu \sin \varphi_0 > 0 \quad (5)$$

The rotation is asymptotically stable and inseparable with  $\varphi_0 = -\arccos(-\gamma/\mu)$ , while unstable rotation with  $\varphi_0 = \arccos(-\gamma/\mu)$  is inseparable only if  $\mu < \sqrt{1/4 + \gamma^2}$ .

## Approximate solution ( $\varepsilon \neq 0$ )

We represent the solution as the series  $\varphi = \tau + \varphi_0 + \varepsilon\varphi_1(\tau) + \dots$

$$\varepsilon^0 : \quad \gamma + \mu \cos \varphi_0 = 0, \quad (6)$$

$$\varepsilon^1 : \quad \ddot{\varphi}_1 + \gamma\dot{\varphi}_1 - \mu \sin(\varphi_0) \varphi_1 = \cos(\varphi_0 + 2\tau), \quad (7)$$

where we take  $\varphi_0 = -\arccos(-\gamma/\mu)$  corresponding to the stable solution of the unperturbed system. Thus, (7) takes the form

$$\ddot{\varphi}_1 + \gamma\dot{\varphi}_1 + \sqrt{\mu^2 - \gamma^2} \varphi_1 = \cos(\varphi_0 + 2\tau) \quad (8)$$

and has the unique periodic solution

$$\varphi_1(\tau) = C \sin(\varphi_0 + 2\tau) + D \cos(\varphi_0 + 2\tau) \quad (9)$$

where

$$C = \frac{2\gamma}{3\gamma^2 + \mu^2 - 8\sqrt{\mu^2 - \gamma^2} + 16}, \quad D = \frac{-4 + \sqrt{\mu^2 - \gamma^2}}{3\gamma^2 + \mu^2 - 8\sqrt{\mu^2 - \gamma^2} + 16}$$

## Stability analysis

We add a small perturbation  $\varphi = \varphi_* + u$  to the true solution  $\varphi_*$  of (1) and linearize (1) w.r.t.  $u$  obtaining the Mathieu-Hill equation with damping

$$\ddot{u} + \gamma\dot{u} + \left( \sqrt{\mu^2 - \gamma^2} + \varepsilon\Phi(2\tau) \right) u = 0, \quad (10)$$

where  $\Phi = (\gamma C + 1) \sin(2\tau + \varphi_0) + \gamma D \cos(2\tau + \varphi_0) + O(\varepsilon)$ .  
Stability condition (absence of parametric resonance in (10))

$$\varepsilon < \frac{2\gamma}{\sqrt{(\gamma C + 1)^2 + \gamma^2 D^2}} + o(\varepsilon) \quad (11)$$

is also the stability condition for the original equation (1) according to the Lyapunov's theorem.

## Non-separability condition in the first approximation

### Solution

$$\varphi = \tau - \arccos\left(-\frac{\gamma}{\mu}\right) + \varepsilon C \sin(\varphi_0 + 2\tau) + \varepsilon D \cos(\varphi_0 + 2\tau) + o(\varepsilon)$$

We substitute it into condition (2) that the hula-hoop is not separated from the waist

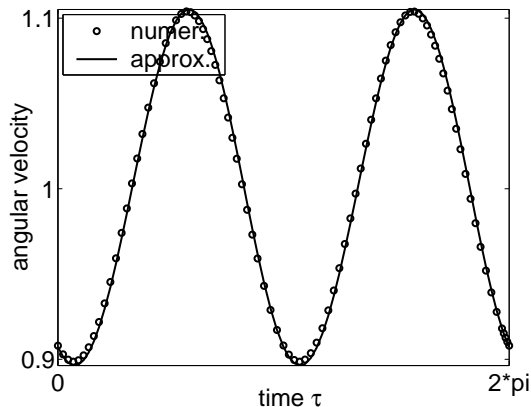
$$\dot{\varphi}^2 - 2\mu \sin(\varphi - \tau) + 2\varepsilon \cos(\varphi + \tau) > 0,$$

which is guaranteed for all  $\tau$  if the following inequality is satisfied

$$\varepsilon < \frac{1 + 2\sqrt{\mu^2 - \gamma^2}}{2} \sqrt{\frac{\mu^2 + 3\gamma^2 - 8\sqrt{\mu^2 - \gamma^2} + 16}{\mu^2 + 8\gamma^2 - 12\sqrt{\mu^2 - \gamma^2} + 36}} + o(\varepsilon).$$



# Comparison between approximate and numerical solutions



Angular velocity  $\dot{\varphi}$   
first approximation (solid line)  
numerical simulation (circles).

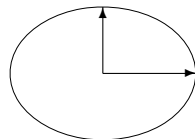
$$\varepsilon = 0.2$$

$$\mu = 1.2$$

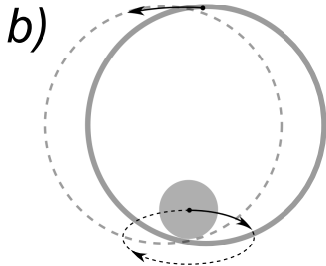
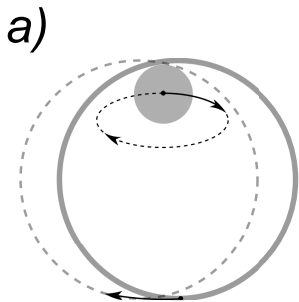
$$\gamma = 1.0$$

$$\frac{a}{2(R-r)} = 1.4$$

$$\frac{b}{2(R-r)} = 1.0$$



If  $\mu$  and  $\gamma$  are also small of order  $\varepsilon$  then both direct and reverse rotation are possible



## Approximate solution when $\mu$ and $\gamma$ are small

We represent the solution as the series

$$\varphi = \rho\tau + \varphi_0 + \varepsilon\varphi_1(\tau) + \dots \quad (12)$$

We introduce new not small parameters  $\tilde{\mu} = \mu/\varepsilon$ ,  $\tilde{\gamma} = \gamma/\varepsilon$ .  
First adjustment  $\varphi_1$  is defined by the equation:

$$\ddot{\varphi}_1 = \cos(\rho\tau + \varphi_0 + \tau) - \tilde{\mu} \cos(\rho\tau + \varphi_0 - \tau) - \tilde{\gamma}\rho \quad (13)$$

By equating separately constant and oscillation terms we find that solutions exist only if angular velocity  $\rho \in \{1, 0, -1\}$ .

With  $\rho = 1$  we get the first order adjustment

$$\varphi_1(\tau) = -\frac{1}{4} \cos(\varphi_0 + 2\tau), \quad \cos \varphi_0 = -\frac{\gamma}{\mu}$$

## Stability and non-separability conditions with small $\mu$ и $\gamma$

Approximate solution with  $\rho = 1$

$$\varphi = \tau + \varphi_0 - \frac{\varepsilon}{4} \cos(\varphi_0 + 2\tau) + o(\varepsilon), \quad \cos \varphi_0 = -\frac{\gamma}{\mu} \quad (14)$$

Stability conditions in first approximation

$$0 < \gamma < \mu, \quad \sin \varphi_0 < 0 \Rightarrow \varphi_0 = -\arccos\left(-\frac{\gamma}{\mu}\right) \pmod{2\pi}$$

Condition that the hula-hoop is not separated from the waist

$$\varepsilon < \frac{1 + 2\sqrt{\mu^2 - \gamma^2}}{3} + o(\varepsilon). \quad (15)$$

## Stability and non-separability conditions with small $\mu$ и $\gamma$

Approximate solution with  $\rho = -1$

$$\varphi = -\tau + \varphi_0 + \frac{\mu}{4} \cos(\varphi_0 - 2\tau) + o(\varepsilon), \quad \cos \varphi_0 = -\frac{\gamma}{\varepsilon} \quad (16)$$

Stability conditions in first approximation

$$0 < \gamma < \varepsilon, \quad \sin \varphi_0 > 0 \Rightarrow \varphi_0 = \arccos\left(-\frac{\gamma}{\varepsilon}\right) \pmod{2\pi}$$

Condition that the hula-hoop is not separated from the waist

$$\mu < \frac{1 + 2\sqrt{\varepsilon^2 - \gamma^2}}{3} + o(\varepsilon). \quad (17)$$

## Condition of coexistence of direct and reverse rotations

Condition of coexistence direct and reverse rotations

$$0 < \gamma < \min\{\varepsilon, \mu\} \quad (18)$$

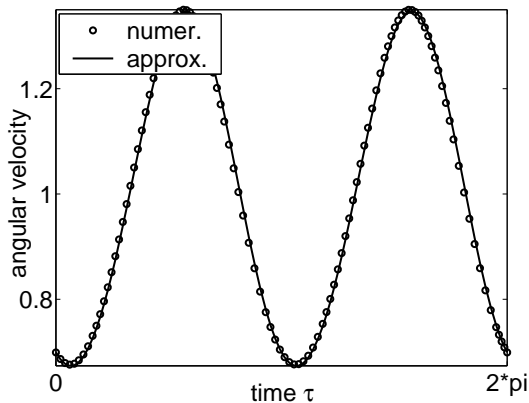
is obtained from combination of  $0 < \gamma < \mu$  (for direct rotation) and  $0 < \gamma < \varepsilon$  (for reverse rotation), where both parameters  $\varepsilon$  and  $\mu$  are supposed to be small.

In physical variables (18) takes the form

$$0 < 2k \frac{R - r}{R^2 \omega m} < a - |b| \quad (19)$$

i.e. the trajectory of the waist center should be sufficiently prolate.

## Angular velocity of direct rotation



Angular velocity  $\dot{\varphi}$   
first approximation (solid line)  
numerical simulation (circles).

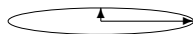
$$\varepsilon = 0.6$$

$$\mu = 0.8$$

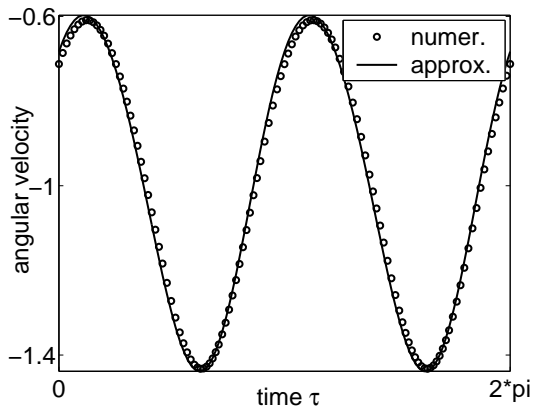
$$\gamma = 0.5$$

$$\frac{a}{2(R-r)} = 1.4$$

$$\frac{b}{2(R-r)} = 0.2$$



## Angular velocity of reverse rotation



Angular velocity  $\dot{\varphi}$   
first approximation (solid line)  
numerical simulation (circles).

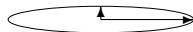
$$\varepsilon = 0.6$$

$$\mu = 0.8$$

$$\gamma = 0.5$$

$$\frac{a}{2(R-r)} = 1.4$$

$$\frac{b}{2(R-r)} = 0.2$$





## Conclusion

- ▶ Exact solutions for the hula-hoop under a circular excitation are obtained and their stability is studied
- ▶ Approximate solutions for an elliptic excitation are found
- ▶ The non-separability condition of the hula-hoop from the waist of a gymnast during rotation is derived
- ▶ The coexisting rotations for the direct and reverse rotations of the hula-hoop are analyzed
- ▶ The analytical solutions are compared with the results of numerical simulation



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*How to twirl a hula hoop,*

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