Excitation of oscillations by a small limited control force

Anton O. Belyakov,
Institute of Mechanics, Moscow State Lomonosov University,
Michurinsky pr. 1, 119192 Moscow, RUSSIA
belyakov@imec.msu.ru

***Abstract*— Linear oscillator with limited excitation force (control) is under consideration. The optimal control, which led oscillatory system to a certain energy level from any initial conditions at minimum time, is found. The control synthesis is made. I. e. time in control function is excluded by current system phase variables (coordinate and velocity). Quasi-optimal synthesized control function is obtained for one-dimensional oscillatory system with unknown parameters. Multidimensional case is considered on the supposition that excitation forces are small.**

I. Optimal control in one-dimensional case

Let as consider a one-dimensional linear oscillator with the limited control force u^*

$$\ddot{x}_1 + \omega^2 x_1 = m^{-1}u^*, \qquad |u^*| \le u_m^*, \tag{1}$$

where x_1 is the scalar variable, m is the mass, and ω is the frequency of free vibrations. It is necessary to find a function u^*, which brings oscillatory system (1) with the initial conditions $x_1(0)$, $\dot{x}_1(0)$ to the given energy level characterized by the amplitude a at the minimal time.

Let us rewrite equation (1) in the Cauchy form

$$\dot{\vec{x}} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ \omega u \end{pmatrix}, \quad \vec{x}(0) = \vec{x}_0, \tag{2}$$

where the vector $\vec{x} = (x_1,\ \omega^{-1}\dot{x}_1)^T$, and the function $u = m^{-1}\omega^{-2}u^*$. The constancy condition of the mechanical energy with oscillation amplitude a gives the equation of the surface in the phase space, to which the system needs to be brought

$$\Phi(\vec{x}) = a^2 - x_1^2 - x_2^2 = 0. \tag{3}$$

We will solve the problem of optimal control by Pontryagin's maximum principle for the case where time is the objective functional [1]. For this purpose we consider the system adjoint to (2)

$$\dot{\vec{\psi}} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \vec{\psi},$$

which has the solution

$$\begin{aligned} \psi_1 &= C_\psi \sin(\omega t + \varphi), \\ \psi_2 &= C_\psi \cos(\omega t + \varphi). \end{aligned} \tag{4}$$

The Hamiltonian of the problem has the form

$$H = \psi_0 + \omega u \psi_2 + \omega x_2 \psi_1 - \omega x_1 \psi_2 \equiv 0, \tag{5}$$

where $\psi_0 < 0$. The necessary optimality conditions for the control function u are given by the Pontryagin maximum principle. It is maximum condition

$$\max_u H \tag{6}$$

with the transversality conditions

$$\psi_i(T) = c\, \frac{\partial \Phi}{\partial x_i}\bigg|_{t=T}, \tag{7}$$

where T is the final time moment.

In the problem under consideration we have from the maximum condition (6)

$$u = u_m \mathsf{sign}\,(\psi_2)\,, \tag{8}$$

with $u_m = m^{-1}\omega^{-2}u_m^*$. The transversality conditions (7) give

$$\begin{aligned} \psi_1 &= c x_1, \\ \psi_2 &= c x_2. \end{aligned} \tag{9}$$

Now we can express the optimal control via phase variables using the conditions of maximum (8) and transversality (9) along with the solution of the adjoint problem (4). Is means that we produce synthesis of control.

A. The synthesis of control

Synthesized control function u does not depend on time but it depends on the phase variables (coordinate and velocity). Hence it is valid for any initial conditions and brings system to the required position regardless of fluctuations. This is more useful in practice and is always desirable.

Optimal control is equal either to u_m or $-u_m$ due to the maximum condition (8). Thus, the optimal trajectory of system (2) on the phase plane consists of arcs of concentric circles with the centers at the points $(u_m, 0)$ or $(-u_m, 0)$. A set of points where the trajectory becomes an arc of a circle with the other center is called the switching line. To synthesize the optimal control means to find the switching line and to determine the sign of the control function u on both sides of the switching line.

The case when $a = 0$ is a classical example of the problem of optimal control with time as the objective functional considered by D.W. Bushaw [2], [1]. The switching line and the optimal trajectory for $a = 0$ are shown in Figure 1. To draw the switching line for $0 < a < u_m$ (see Figure 2) we note from adjoint system solution (4) that between

0-7803-9235-3/05/$20.00 © 2005 IEEE

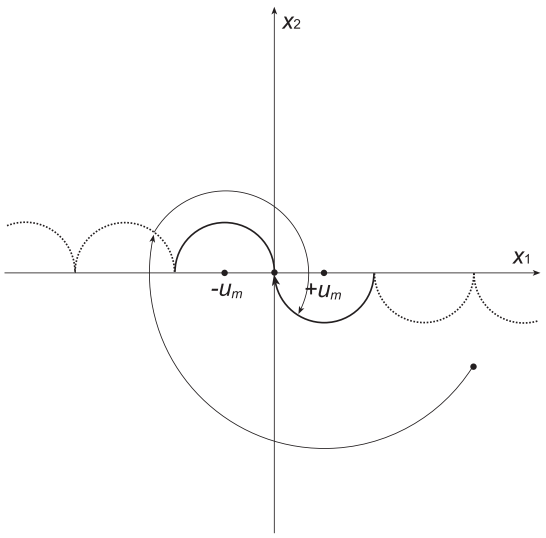

Fig. 1. Switching line and optimal trajectory example for $a = 0$

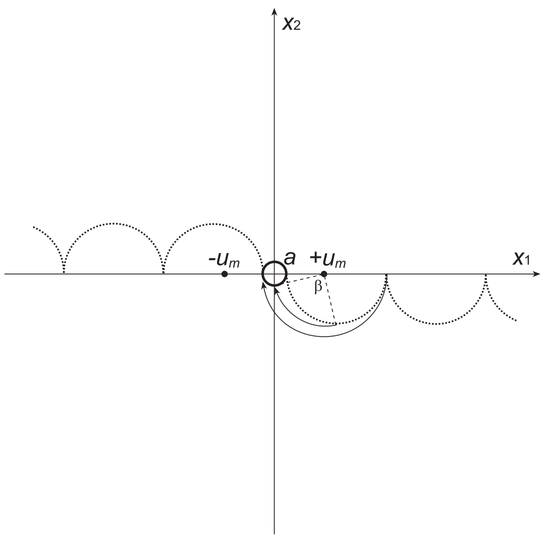

Fig. 2. Switching line for $0 < a < u_m$

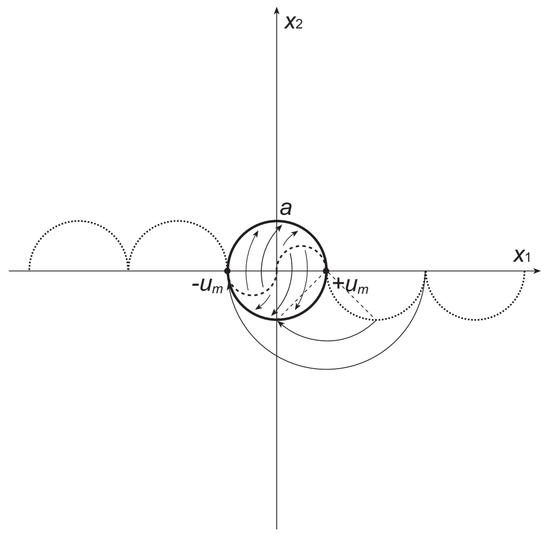

Fig. 3. Switching line for $a = u_m$

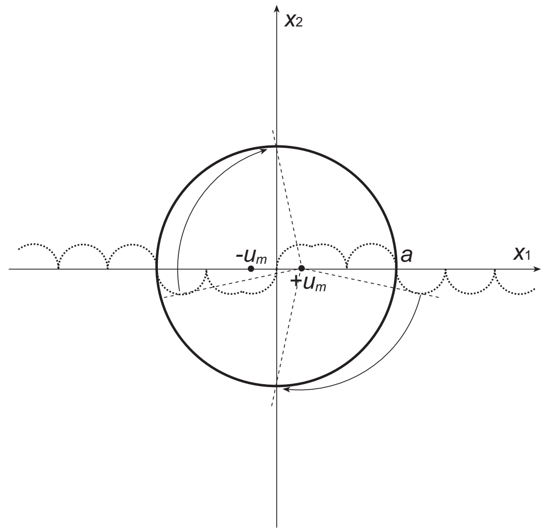

Fig. 4. Switching line for $a > u_m$

two switches the affix must pass the angle π on the same circumference around it's center. Thus, the switching line is symmetric with respect to the origin. Considering the process in the inverse time from the terminal surface (3) we can see from the transversality conditions (9) and solution of the adjoint problem (4) where should the first (in inverse time) switch occur depending on the initial point on the terminal surface (3).

Let us find an angle β after passing which the first (in inverse time) switch occurs. From solution (4) for the adjoint system we get

$$\sin(\beta) = \frac{\psi_2(T)}{\sqrt{\psi_1^2(T) + \psi_2^2(T)}} \tag{10}$$

Using transversality conditions (9) this formula can be expressed through x_1 and x_2

$$\sin(\beta) = \frac{c\ x_2(T)}{|c|\sqrt{x_1^2(T) + x_2^2(T)}}. \tag{11}$$

After taking into consideration the terminal surface equation (3) we find

$$\sin(\beta) = \pm\ x_2(T)/a. \tag{12}$$

Thus, noting that in inverse time the affix must move counterclockwise (i. e. $\beta > 0$) we get

$$\beta = \arcsin(|x_2(T)|/a). \tag{13}$$

For instance, it is clearly seen that if we start from the point $(0, -a)$ the control will be switched after passing the angle $\pi/2$ around the point $(u_m, 0)$. For drawing switching line near the origin (see for example Figure 3) we note from the system (1) that time is proportional to the sum of the angles

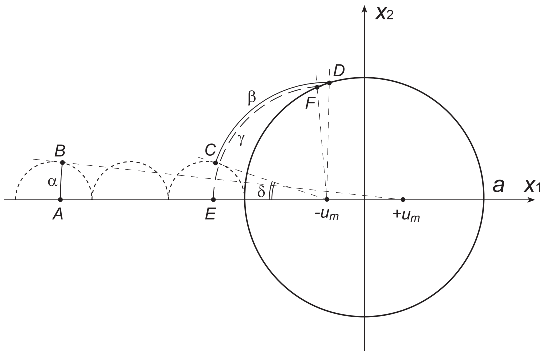

Fig. 5. Optimal and quasi-optimal trajectories for $a > u_m$.

passed by affix on the arches. Similarly, the switching line $a > u_m$ is obtained (see Figure 4).

B. The synthesis of control at $a \gg u_m$ with unknown system parameters

In practice it is often impossible to gain frequency of a system whose oscillations should be caused by the control u. For example, it is impossible when the problem is to find the frequency of the system from its oscillations. Even the exact value of the control bound u_m cannot always be provided. Therefore, we can't draw the switching line as we did earlier.

However, when $a \gg u_m$ the abscissa axis can be taken as a switching line (see Figure 4). The control function

$$u = \begin{cases} u_m \mathsf{sign}\,(x_2) & \text{when } x_1^2 + x_2^2 < a^2 \\ 0 & \text{when } x_1^2 + x_2^2 = a^2 \\ -u_m \mathsf{sign}\,(x_2) & \text{when } x_1^2 + x_2^2 > a^2 \end{cases}$$

in this case takes the form

$$u = -u_m \mathsf{sign}\,(\Phi(\vec{x}))\,\mathsf{sign}\left(\frac{\partial \Phi}{\partial x_2}\right). \qquad (14)$$

In previous sections we obtained optimal control. The control function (14) is not optimal. But it is the closer to optimal the greater a than u_m (see Figure 4).

C. Maximal time of the system excitation

Now we can estimate the time T for the case when $a \gg u_m$ and $x_1(0) = \dot{x}_1(0) = 0$. It is easy to see from Figure 4 that during one period the amplitude of the system oscillations is changed at the quantity $4\ U_m$. Thus, dividing a by $4\ U_m$ and multiplying by period $2\pi/\omega$ we get

$$T = \frac{\pi\ a}{2\ u_m\ \omega}. \qquad (15)$$

After substitution of $u_m = m^{-1}\omega^{-2}u_m^*$ finally we obtain

$$T = \frac{\pi\ a\ m\ \omega}{2\ u_m^*}. \qquad (16)$$

Let as determine how time (16) differs from optimal time. On Figure 5 we can see the differences between the trajectories on phase plane for optimal control (solid line) and for quasi-optimal control (14) (dash line). As for the time of the process is proportional to the sum of angles that affix passes on arcs around different centers let as compare this sums for optimal and quasi-optimal processes. Without loss of generality we can consider processes beginning from point A. Optimal trajectory passes angle α on the arc around point $(u_m, 0)$, then the trajectory makes n turns by π in tern around $-(u_m, 0)$ and $(u_m, 0)$ and finally passes angle β from point C to point D placed on the terminal surface. Thus we have the following angle sums

$$\begin{aligned} \alpha + 2\pi n + \beta \quad &- \quad \text{optimal process,} \\ 2\pi n + \gamma \quad &- \quad \text{quasi-optimal process.} \end{aligned}$$

As for the optimal process has minimal angle sum we can write the following inequality

$$\gamma \geqslant \alpha + \beta \qquad (17)$$

Besides, we can see from the Figure 5, that point D is placed left from point F on the terminal surface, thus the following inequality is valid

$$\beta \geqslant \gamma - \delta.$$

Adding to the left side of this inequality nonnegative angle α and taking into consideration inequality (17) we get

$$\gamma \geqslant \alpha + \beta \geqslant \gamma - \delta.$$

At $a \gg u_m$ angle $\delta \sim u_m/a$, therefore we can write the following

$$\gamma = \alpha + \beta + O(u_m/a).$$

Adding the angle $2\pi n$ to both sides of the latter equation and dividing them by ω, we obtain the following expression for the time of quasi-optimal process

$$T = T_0 + \frac{O(u_m/a)}{\omega},$$

where T_0 is the time of optimal process. Thus the time (16) can be considered as optimal accurate to the value

$$\frac{O(u_m/a)}{\omega}.$$

The same time estimation can be obtained with initial point within the terminal surface but not close to coordinate origin. If the initial point is near to coordinate origin the difference between the quasi-optimal and the optimal process times not exceeds the value π/ω.

In comparison with excitation by sinusoidal force of the same amplitude u_m^*

$$\ddot{x} + \omega^2 x = u_m^* \sin(t)/m, \quad x(0) = \dot{x}(0) = 0 \qquad (18)$$

excitation by force (14) provides the time which is in $4/\pi$ times shorter.

II. Minimal time estimation for excitation of multi-dimensional system

Multi-dimensional linear conservative system is governed by the equation

$$\mathbf{M}\ddot{\vec{x}} + \mathbf{K}\vec{x} = \mathbf{B}^*\vec{u}, \quad \vec{x}(0) = \dot{\vec{x}}(0) = 0, \qquad (19)$$

$$|u_j| \leq U_j, \quad j = 1, \ldots, m\ , \qquad (20)$$

where the matrices $\mathbf{M}$ and $\mathbf{K}$ are symmetrical and positive definite, $\vec{x}$ is the vector of variables of dimension n, $\vec{u}$ is control vector of dimension m, $\mathbf{B}^*$ is $n \times m$ control matrix. The problem is to estimate the minimal time, at which all the modes of system's oscillations can achieve the corresponding energy levels.

It is well known that there is a linear change of coordinates with the matrix $\mathbf{S}$, which transforms the matrix $\mathbf{M}$ to the identity matrix and the matrix $\mathbf{K}$ to a diagonal matrix ω^2 containing the squared natural frequencies ω_i^2 of the system

$$\ddot{\vec{z}} + \omega^2 \vec{z} = \mathbf{B}\vec{u}, \quad \vec{z}(0) = \dot{\vec{z}}(0) = 0, \tag{21}$$

where $\vec{z}$ is the vector of normal coordinates, and $\mathbf{B} = \mathbf{S}^T\, \mathbf{B}^*$. Each of the equations

$$\ddot{z}_i + \omega_i^2 z_i = (\mathbf{B}\vec{u})_i, \quad z_i(0) = \dot{z}_i(0) = 0, \tag{22}$$

coincides with (1) up to notation.

Using time estimation (16) for one-dimensional oscillator (1) we can select a multi-dimensional oscillator mode with maximum excitation time in the case when all control is directed to this mode excitation. Thus, we obtain minimal time estimation for multi-dimensional oscillator

$$T_{min} = \frac{\pi}{2} \max_i \left[a_i \omega_i \Big/ \sum_{j=1}^{m} \left(|B_{ij}| U_j \right) \right], \tag{23}$$

where a_i is the given terminal amplitude of the i-th normal coordinate.

III. Concluding remarks

In the presented paper, on the base of the synthesis of quasi-optimal control for one-dimensional oscillator we have obtained time estimation for excitation of oscillations. Quasi optimal time estimation tends to optimal when ratio of control limitation to the terminal amplitude tends to zero. Then, with this estimation we have derived the minimal excitation time for multi-dimensional oscillator. The obtained result can be used for testing the effectiveness of feedback algorithms for multi-dimensional systems.

References

[1] V.V. Alexandrov, V.G. Boltyanski, S.S. Lemak, N.A. Parusnikov, V.M. Tihomirov. "Optimisation of controlled system dynamics" M.: Isd-vo Mosk. Univ. 2000. 304 p. (in Russian)

[2] D.W. Bushaw. "Experimental towing tank" *Stevens Inst. of Technology.* Preprint 169. N.Y.: Hoboken, 1953.