The Swing Dynamics

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Key questions

- Instability regions for the swing
- Regular and chaotic motions

Methods

- Stability analysis is based on derivatives of the Floquet matrix with respect to problem parameters
- Averaging method
- Numerical simulation

Periodical Systems

Alexander M. Lyapunov
(1857-1918)

Thesis (1892)
"General problem on stability of motion"

Chapter 3
Study of periodic movements

$$\dot{\mathbf{x}} = \mathbf{G}\mathbf{x}$$

Introduction of parameters

$$\mathbf{G}(t, \mathbf{p}) = \mathbf{G}(t + T, \mathbf{p})$$

$\mathbf{p}$ – is the vector of parameters

Stability of periodic systems

General stability theory by Floquet (1883)

Periodic system $\dot{\mathbf{x}} = \mathbf{G}\mathbf{x}$, $\mathbf{G}(t) = \mathbf{G}(t+T)$

Matriciant X(t): $\dot{\mathbf{X}} = \mathbf{G}(t)\mathbf{X}$, $\mathbf{X}(0) = \mathbf{I}$

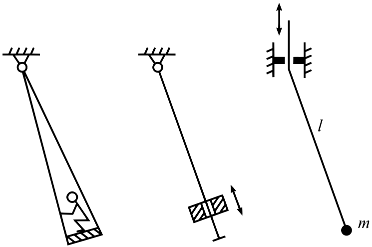
Monodromy matrix $\mathbf{F} = \mathbf{X}(T)$

Multiplier ρ: $\mathbf{F}\mathbf{u} = \rho\mathbf{u}$, $|\rho| < 1$ – *asymptotic stability*; $|\rho| > 1$ – *instability*

Bifurcations of multipliers (Seyranian, Solem, Pedersen (1999) Arch.Appl.Mech.)

$$\frac{\partial \mathbf{F}}{\partial p_j} = \mathbf{F}\int_0^T \mathbf{X}^{-1}\frac{\partial \mathbf{G}}{\partial p_j}\mathbf{X}dt, \qquad \frac{\partial \rho}{\partial p_j} = \rho\mathbf{v}^T\int_0^T \mathbf{X}^{-1}\frac{\partial \mathbf{G}}{\partial p_j}\mathbf{X}\mathbf{u}dt$$

A swing is the simplest model of parametric resonance



Nonlinear system

a pendulum of variable length:

$$\dot{(ml^2\dot\theta)} + \gamma l^2\dot\theta + mgl\sin\theta = 0$$

$$l = l_0 + a\varphi(\Omega t)$$

Resonant frequencies:

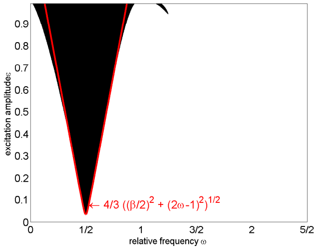
$\Omega_k = 2\Omega_0/k$,

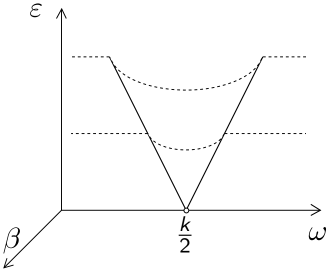
$k = 1, 2, \ldots, \quad \Omega_0 = \sqrt{\frac{g}{l_0}}$

Three non-dimensional parameters:

$$\varepsilon = \frac{a}{l_0}, \quad \omega = \frac{\Omega_0}{\Omega}, \quad \beta = \frac{\gamma}{m\Omega_0}.$$

Instability region for the swing is a half-cone in parameter space $(\varepsilon, \beta, \omega)$





Seyranian (2004) Doklady Physics

$$(\beta/2)^2 + (2\omega/k - 1)^2 < (3\varepsilon/4)^2(a_k^2 + b_k^2), \quad \beta \geq 0, \quad k = 1, 2, \ldots,$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(\tau)\cos(k\tau)d\tau, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(\tau)\sin(k\tau)d\tau.$$

Limit cycle

Non-linear Mathieu-Hill equation for small angles

$$\ddot{q} + \beta\omega\dot{q} + \left[\omega^2 - \varepsilon(\ddot{\varphi}(\tau) + \omega^2\varphi(\tau))\right] q - \frac{\omega^2}{6} q^3 = 0,$$

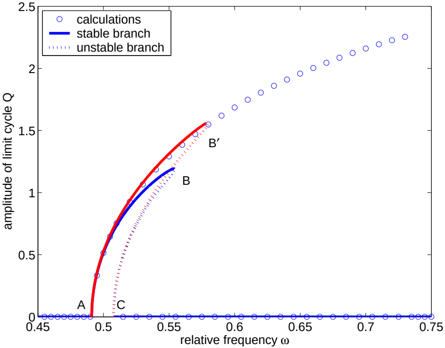
where $q = \theta\,(1 + \varepsilon\varphi(\tau))$, $\varphi(\tau) = \cos\tau$.

Averaging method

$q(\tau) = Q(\tau)\cos(\tau/2 + \Psi(\tau)) + \varepsilon u(\varepsilon, \tau, Q, \Psi)$, where
$\int_0^{4\pi} u(\varepsilon, \tau, Q, \Psi)\sin(\frac{\tau}{2})d\tau = \int_0^{4\pi} u(\varepsilon, \tau, Q, \Psi)\cos(\frac{\tau}{2})d\tau = 0.$

$$\begin{aligned} \dot{Q} &= -\frac{Q\beta\omega}{2} + \frac{Q\varepsilon(1-\omega^2)}{2}\sin(2\Psi), \\ \dot{\Psi} &= \omega - \frac{1}{2} - \frac{Q^2\omega^2}{8} + \frac{\varepsilon(1-\omega^2)}{2}\cos(2\Psi). \end{aligned}$$

Frequency-response curve



Steady motion:

$$\dot{Q} = 0, \quad \dot{\Psi} = 0$$

$$Q^2 = \frac{4}{\omega^2}\left(2\omega - 1 \mp \sqrt{\varepsilon^2(1-\omega^2)^2 - \beta^2\omega^2}\right),$$

$$\Psi = \frac{1}{2}\arctan\left(\frac{\mp 4\beta\omega}{\sqrt{\varepsilon^2(1-\omega^2)^2 - \beta^2\omega^2}}\right) + \pi j,\ j = \ldots, -1, 0, 1, \ldots$$

Stability of the limit cycle

Linearization: $q(\tau) = q_0(\tau) + \delta(\tau)$, where $|\delta(\tau)| \ll 1$,
$q_0(\tau) = Q(\tau)\cos(\tau/2 + \Psi(\tau)) + \varepsilon u(\varepsilon, \tau, Q, \Psi)$

Hill's equation with damping:

$$\ddot{\delta} + \beta\omega\dot{\delta} + \left[\omega^2 + \varepsilon(1-\omega^2)\cos(\tau) - \frac{\omega^2}{2} q_0^2(\tau)\right]\delta = 0.$$

Instability condition

$$\mp Q^2\omega^2\sqrt{\varepsilon^2(1-\omega^2)^2 - \beta^2\omega^2} < 0$$

Regular rotations

$$\ddot{\theta} + \left(\frac{2\varepsilon\dot{\varphi}(\tau)}{1+\varepsilon\varphi(\tau)} + \beta\omega \right) \dot{\theta} + \frac{\omega^2 \sin\theta}{1+\varepsilon\varphi(\tau)} = 0 \quad \Rightarrow \quad \dot{\mathbf{x}} = \varepsilon \mathbf{Z}(\mathbf{x}, \tau)$$

If average rotational velocity exists

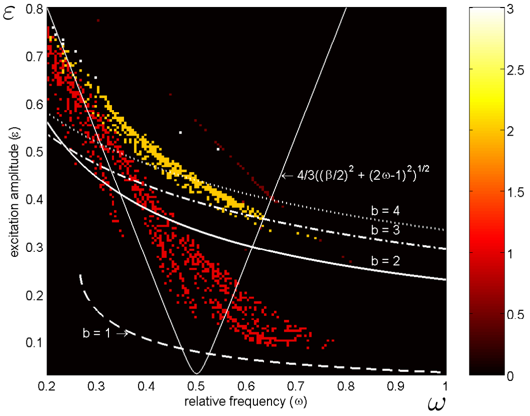
$$b = \lim_{T\to\infty} \frac{1}{T} \int_0^T \dot{\theta} d\tau$$

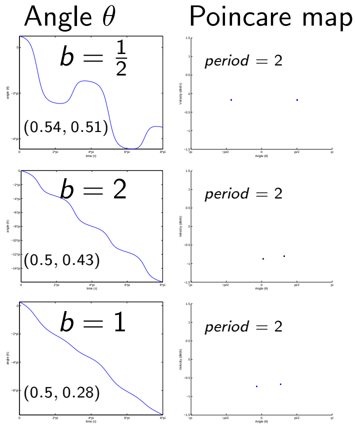
we can introduce vector **x** of *slow variables*

fast time	phase mismatch	velocity	excitation
$s = b\tau$	$x_1 = \theta - s$	$x_2 = \dot{\theta}$	$x_3 = 1 + \varepsilon\cos(s/b)$

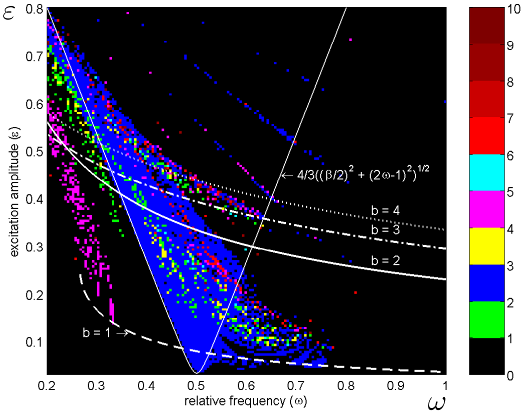
$$\begin{aligned} \dot{x}_1 &= x_2 - 1, \\ \dot{x}_2 &= -\frac{\varepsilon}{b} \frac{2\sin(s/b)}{x_1} x_2 - \beta\omega x_2 - \frac{\omega^2}{b^2} \frac{\sin(x_3+s)}{x_1}, \\ \dot{x}_3 &= -\frac{\varepsilon}{b} \sin(s/b). \end{aligned}$$

Regular rotations ($\beta = 0.05$)

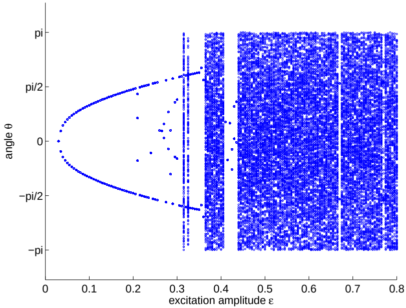




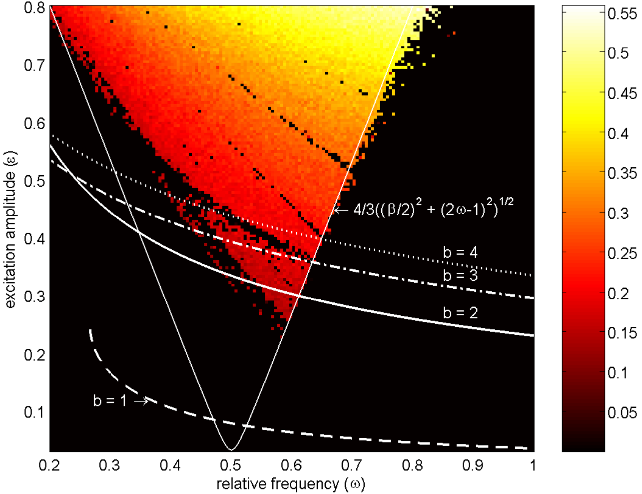
Periods of regular motions ($\beta = 0.05$)



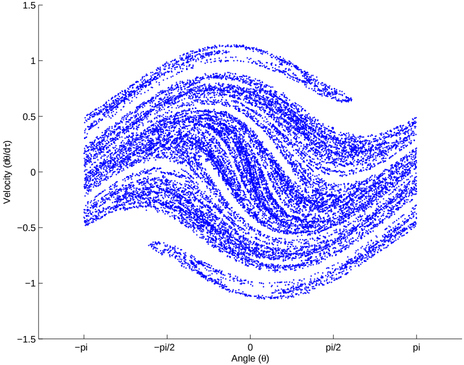
Bifurcational diagram ($\omega = 0.5$)



Lyapunov exponents ($\beta = 0.05$)



Strange attractor



The Poincare map for the parameters $\varepsilon = 0.255$, $\omega = 0.59$, and $\beta = 0.05$.